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MAXIMALLY UNCOUPLED GENERALIZED NODAL EQUATIONS

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TABLE OF CONTENTS

I. INTRODUCTION

A. Problem Statement

For that large class of electrical problems which can be modeled as lumped networks, Kirchhoff's voltage law--which states that the sum of the voltages around any closed loop is zero--and Kirchhoff's current law--which in its generalized form states that the net current outflow from a point or finite group of points is zero--are very fundamental network properties which are the foundations of network analysis. If the voltage-current relationships are known for every branch of the network, the use of the Kirchhoff equations results in the formulation of equations which completely describe the network. Two procedures have been developed for systematically applying the Kirchhoff laws to a network. These two, which differ essentially only in the order in which the Kirchhoff laws are utilized,are known as the loop current method and the node voltage method. For most networks there are many possible choices of independent sets of Kirchhoff equations, hence there are many ways to formulate loop and node equations. The purpose of this investigation is to find formulation procedures which lead to an independent set of network equations which are optimum or near-optimum in the sense of being maximally uncoupled. Since the loop current and node voltage formulations are so interrelated, attention will be restricted to the node voltage formulation only, with the expectation that similar results could be found for the loop current formulation.

The problem to be investigated, simply stated, is this. Given an interconnection diagram for a lumped electrical network, how can one write an independent set of equations in which the variables are linear combinations of node voltages in a manner such that the number of zeros in the coefficient matrix is maximized? Note that the problem differs from a normal co-ordinate (eigenvector) problem in that it matters only whether or not there is a current path between a pair of nodes, with no use being made of the admittance value of this path. The class of electrical networks to which this investigation applies is limited to those finite lumped networks which can be represented as an interconnection of two-terminal elements having no closed loops of infinite admittance elements (short circuits). The practical effect of these restrictions is to rule out unsymmetrical mutual couplings and perfectly coupled transformers. The method of attack on the problem is to decompose the coefficient matrix of the nodal equations into constituent matrices by the use of linear graph theory.

B. Establishing the Graph Model of the Problem

Busacker and Saaty (1) formally define an abstract directed graph to consist of a nonempty set V, a set E disjoint from V, and a mapping of E into the cartesian product VxV. Since a graph is an abstract entity, a correspondence must be chosen between its elements and the elements of the physical problem under consideration. The correspondence chosen is to relate the nodes of the electrical network to the elements of the set V and the current paths of the network to the elements of the set E. This choice is a very simple and natural one because the geometric graph isomorphic to the abstract graph defined above has the appearance of a skeleton of the electrical network. This particular choice of correspondence is the reason that the class of electrical networks studied was restricted to those reducible to an interconnection of two-terminal elements in a manner that no closed loops of infinite admittance elements result. Parallel circuit elements will be considered to be combined into a single current path, so the corresponding graph will have no parallel edges. Also, self-loops have no meaning in an electrical circuit and will not appear in the corresponding graph, thus the graph to be used is restricted to have no parallel edges and no self-loops. Busacker and Saaty (1) term such a graph a "simple" directed graph.

In order that a consistent reference convention be followed in relating the voltage and current in the individual two-terminal elements, it is necessary to use a directed rather than a non-directed graph. Since the network has been restricted to be composed of two-terminal components only, every pair of vertices which are joined by a directed edge with admittance weight y_{ij} are also joined by an oppositely directed edge of weight $-y_{ij}$. Note that the edge orientations have nothing to do with direction of current flow but simply determine the signs of the weight factors to be used in a graph operation such as describing a path or disconnecting set. Since the graph to be used contains oppositely directed edges with weights of equal magnitude but opposite sign between every adjacent pair of vertices, it is conventional in the electrical engineering literature to draw the geometric graph with a single directed edge joining adjacent vertices. It is then understood that in an operation with an orientation confluent to the edge orientation the weight factor is used with its given sign, and in an operation with orientation counterfluent to the edge orientation, the weight factor is used with the negative of its given sign.

A graph is said to be separable if it contains an articulation point. An articulation point of a connected graph G is a vertex such that the subgraph obtained by deleting this vertex and all edges incident upon it from G is disconnected. Separable graphs and disconnected graphs (including isolated vertices) are not very interesting from a nodal equation point of view because the formulation of equations for each separated or separable part can be handled as a problem not related to the other graph parts. Restricting the problem to connected nonseparable graphs is not really a restriction, however, since then an optimum overall matrix can always be written as a direct sum matrix containing the optimum matrices for each separable or separated part.

The major graph-theoretic concept of use in this investigation is that of a vertex segregation or seg. Some of the properties of segs are discussed in Reed (3) and in Reed (4). A seg is defined as follows. Given a graph with the set of vertices V, partition the vertices into two all-inclusive mutually-exclusive non-empty sets w and $\overline{w}$. Then the set of all edges joining a vertex of w to a vertex of $\overline{w}$ constitutes a seg. Many of the useful properties of a seg of a graph do not depend on the edge structure of the graph, hence it is sometimes convenient to consider the seg to be a set of vertices. A knowledge of the elements of either vertex set w or vertex set $\overline{w}$ for a given graph is sufficient to completely describe a seg. Any seg for which one of the partitioned vertex sets consists of a single vertex is referred to as a star, and all segs which are not stars will be referred to as multiple segs. Any two segs corresponding to partitions which cannot be drawn on a geometric graph without crossing over one another are defined to be interlocking segs. A more formal definition is as follows. Let w_i, $\overline{w}_i$, w_j, and $\overline{w}_j$ be the vertex sets generated by segs q_i and q_j respectively. Then q_i and q_j are said to be interlocked if and only if neither w_i nor $\overline{w}_i$ is a proper subset of either w_j or $\overline{w}_j$.

Another concept closely allied to that of vertex segregation is that of cut-set. The term cut-set, as commonly used in the electrical engineering literature, refers to a set of edges which is a special case of a seg. A cut-set is a set of edges which is a seg and which if removed from a graph increases the number of connected parts of the graph by one. Thus if a graph is connected, a set of edges corresponding to a seg is also a cut-set only if its removal from the graph separates the graph into precisely two connected parts. Another viewpoint is that a seg is always a cut-set or an edge-disjoint union of cut-sets.

A complete seg matrix Q_a has a row for each possible seg of a graph and a column for each edge. Each seg is assigned an arbitrary orientation, and the matrix elements q_{ij} are assigned values as follows:

q_{ij} = 1 if edge j is in seg i and their orientations agree,

q_{ij} = -1 if edge j is in seg i and their orientations are opposite,

q_{ij} = 0 if edge j is not in seg i.

For a graph with n vertices, the complete seg matrix Q_a will have $2^{n-1}-1$ rows corresponding to the number of different ways n objects can be partitioned into two non-empty sets. If the graph is connected, the rank of Q_a is always n-1. This is established in Seshu and Reed (6) for cut-sets (their Theorem 5-14), and the proof carries over directly for segs. A very important set of submatrices of Q_a consists of those matrices obtained by deleting all but n-1 rows of Q_a in a manner such that the resulting matrix is of rank n-1. Such matrices will be called simply "seg matrices". A graph with n vertices will be described by $\frac{1}{(n-1)!}\prod_{p=0}^{n-2}(2^{n-1} - 2^p)$ different seg matrices, where different matrices means matrices not derivable from one another by permutations or row or column sign changes. Every cut-set matrix of a graph is also a submatrix of Q_a, but the number of cut-sets of a graph depends on the edge structure as well as the number of vertices of a graph. Another important submatrix of Q_a is A_a, the matrix composed of those n rows which correspond to stars. If the segs of this matrix are all assigned orientations from the smaller to the larger vertex sets, the resulting matrix is known as the complete vertex incidence matrix for a graph, and any of the n-1 matrices formed by deleting a single row from the complete vertex incidence matrix will be referred to as simply "incidence matrices" and denoted by the letter A. The vertex corresponding to the deleted row will be termed the "reference vertex". Still another important set of submatrices of Q_a is composed of those seg matrices for which every seg is the only seg partitioning some vertex pair of the graph. Such matrices will be referred to as "fundamental seg matrices". A fundamental cut-set matrix is defined to be a cut-set matrix for which every cut-set contains an edge not contained in any other cut-set of the matrix. A fundamental cut-set matrix is thus a special case of a fundamental seg matrix, one in which every vertex pair separated by only one seg is an adjacent pair.

Seshu and Balabanian (5) and Seshu and Reed (6) show that a set of generalized nodal equations can be written as $QYR^TV_n = QI$ where Q and R are cut-set matrices, Y is the branch admittance matrix, V_n is the node variable matrix, and I is the source matrix. Their development is unaffected if Q and R are considered to be seg rather than cut-set matrices. Since only networks composed of two-terminal elements will be considered in this study, Y can be assumed to be diagonal. For this condition it is not necessary to actually perform the matrix multiplications of QYR^T to form the coefficient matrix, for the elements of the coefficient matrix may be written directly by inspection. The procedure for accomplishing this is given in the Appendix. The elements of the column matrix V_n will be node variables, that is, linear combinations of node voltages, and this matrix can in most cases be written by inspection. Techniques for determining the entries in V_n will also be included in the Appendix.

Since Y is by hypothesis a diagonal matrix, it will weight the nonzero entries of QYR^T but will not affect the number of zero entries for the general case of unspecified element admittances. Hence we need only study QR^T. Every zero entry in QYR^T will also be zero in QR^T, but because leaving Y out of the matrix product is equivalent to assigning all the admittance weights equal values, cancellations can occur resulting in QR^T having zeros not present in QYR^T. This presents no problem for future developments, but to ensure that no confusion arises those zeros of QR^T which are also zeros of QYR^T for any diagonal Y will be termed "essential zeros" while those arising from cancellations of specific admittances will be termed "non-essential zeros" whenever a possibility of ambiguity is present.

The matrix product QR^T has a useful vector interpretation. Q and R can both be considered to be sets of segs which are bases for the complete set of segs which can be associated with a graph, and every zero entry in QR^T corresponds to a seg of Q being orthogonal to a seg of R in the scalar product sense. A somewhat more restrictive definition of seg orthogonality is necessary if only essential zeros are to be considered. For this reason two segs will be defined to be orthogonal if and only if for every nonzero entry in one the corresponding entry in the other is zero. This definition is equivalent to stating that any pair of segs which do not have a common edge are orthogonal. Conversely, a pair of segs which have a common edge are said to be adjacent. This implies that every seg of a connected graph is adjacent to itself. The orthogonality of a basis set of segs is defined to be the number of orthogonal pairs of segs in that set, and the adjacency of a basis set of segs is defined to be the number of adjacent pairs of segs in that set. Thus twice the orthogonality of a basis set Q gives the number of essential zero entries in QQ^T and twice the adjacency of Q gives the number of nonzero and non-essential zero entries.

The matrix $A_aYA_a^T$ is known as the indefinite admittance matrix and has many interesting properties and uses. A discussion of this matrix may be found in Huelsman (2).

C. Preliminary Results

It is interesting to ponder on how many essentially different sets of nodal equations can be written for an unspecified network with n nodes under the restrictions which have been assumed. This is given by the square of the number of different seg matrices which can be written for the graph, and so is a very large number. For example, for n as small as n = 6 there are nearly seven billion different sets of nodal equations describing the network. Thus it is not normally feasible to catalog all different sets of nodal equations for a network as a means of finding an optimum set. The number of different seg matrices is finite for any given graph, however, so the existence of at least one maximally uncoupled set of equations is assured. In the special case that the two seg matrices used in formulating the equation set are equal, a seg matrix yielding maximally uncoupled equations will be a basis set of segs containing a maximum number of orthogonal pairs of segs.

Consider a complete graph, that is, a graph in which every vertex is adjacent to every other vertex. Since every vertex or group of vertices is connected to every other vertex for such a graph, no orthogonal pairs of segs can be found. Thus the $Q\underline{Y}R^T$ coefficient matrix can have no essential zeros and every seg matrix represents a maximally orthogonal set of segs. If a single edge is removed from a complete graph, the resultant graph has precisely one orthogonal seg pair. This seg pair is the two stars corresponding to the two vertices upon which the removed edge had been incident. In this case the coefficient matrix can have a maximum of two zeros and this maximum will be achieved if both Q and R contain the two stars referred to above. Any seg matrix containing these two stars will then represent a maximally orthogonal basis set of segs for this graph. The continuation of this argument suggests, and it will later be proved, that an upper bound on the number of essential zeros achievable in a coefficient matrix for generalized nodal equations for a connected graph is given by twice the number of edges by which the graph fails to be a complete graph.

II. THEOREMS AND PROOFS

As an organizational convenience, the formal statements of theorems and their proofs will be given in this chapter, but a discussion of their implications will be deferred to the succeeding chapter.

Theorem 1. Every vertex pair of a graph must be partitioned by at least one seg of a basis set.

Proof. A basis set of segs for an n-vertex graph G consists of n-1 segs. If a vertex pair is not separated by any seg of a set of n-1 segs, coalesce that vertex pair to form a new graph G'. This will not affect the set of n-1 segs, but since G' has n-1 vertices, at most n-2 of the set of segs can be independent, and so the set cannot be a basis for G.

Theorem 2. If a member of a basis set of segs is the only seg partitioning some vertex pair of a graph, then every other vertex pair partitioned by that seg must also be partitioned by at least one other seg.

Proof. Suppose a set of n-1 segs is given for a graph G, one seg of which is the only seg separating two vertex pairs of G. Coalesce both vertex pairs to form a graph G' with n-2 vertices and delete the seg referred to above from the set. The remaining segs are unaffected by this operation, but now a set of n-2 segs is written for a graph with n-2 vertices. Thus the remaining segs cannot all be independent, and the original set cannot be a basis.

Theorem 3. A set of n-1 segs of an n-vertex graph is independent if and only if it is reducible to an incidence matrix by means of a nonsingular transformation. That is, a matrix Q whose rows correspond to segs is a seg matrix if and only if Q = DA where A is an incidence matrix and D is nonsingular.

Proof. The rank of A is n-1 by definition, and since D is nonsingular, the rank of Q is n-1, hence Q is a seg matrix. Conversely, if Q is a seg matrix it has rank n-1. But A is also a seg matrix, so Q and A must be related by some nonsingular transformation matrix D.

Theorem 4. If a multiple seg q_i is independent of a set of segs F, there exist at least two other segs q_j and q_k which are also independent of F and are such that w_j is a proper subset of w_i and w_k is a proper subset of $\overline{w}_i$, where w_j is a vertex set generated by q_j, w_k is a vertex set generated by q_k, and w_i and $\overline{w}_i$ are the two complementary vertex sets generated by q_i.

Proof. The two vertex sets generated by a multiple seg each contain two or more vertices, hence either may be subdivided. Then q_i may be written as the sum of the segs corresponding to the subdivision, and at least one of these must be independent of F if q_i is. Subdividing the other vertex set yields another seg independent of F in similar fashion.

Corollary 4a. If a star q_{vi} is independent of a set of segs F, there exists at least one other seg q_l which is also independent of F and is such that w_l is a proper subset of $\overline{w}_{vi}$, where w_l is a vertex set generated by q_l and $\overline{w}_{vi}$ is the vertex set generated by q_{vi} which does not consist of a single vertex.

Proof. This is a special case of Theorem 4 in which only one of the vertex sets generated by the given seg has any proper subsets.

Corollary 4b. If a multiple seg q_i is independent of a set of segs F, there exist at least two stars q_{vj} and q_{vk} which are also independent of F.

Proof. Theorem 4 can be applied repeatedly until single vertex sets are achieved.

Theorem 5. If a graph containing n vertices has a vertex of degree n-1, the incidence matrix A for which the maximum degree vertex is reference is such that AA^T has the maximum possible number of essential zeros for a seg matrix product.

Proof. Let Q and R be seg matrices such that QR^T is an optimum coefficient matrix. Suppose a seg q_i of the set composing Q is orthogonal to segs r_j, r_k, ..., and r_m of R. Each seg partitions the vertices of the graph into two disjoint sets, one of which contains the reference vertex and one of which does not. Denote the latter vertex sets by w_{qi}, w_{rj}, ..., and w_{rm} where the subscripts indicate the corresponding seg. The intersection of w_{qi} with any of w_{rj}, w_{rk}, ..., or w_{rm} is void, for if w_{qi} has a vertex in common with, say, w_{rj}, then q_i and r_j must have a common edge, the edge joining the common vertex to the reference vertex. Thus q_i can be written as a sum of stars, every one of which is orthogonal to every seg to which q_i is orthogonal. By Corollary 4b, at least one of these stars must be independent of the remaining segs of Q, so q_i can be replaced by this star with no loss in orthogonality. The process can then be applied to every seg in Q and in R until both Q and R have been reduced to A.

Theorem 6. Given a connected graph G, an upper bound on the number of possible essential zero entries in any nodal equation coefficient matrix is given by the number of zero entries in $A_aA_a^T$, where A_a is the complete vertex incidence matrix for the graph.

Proof. Consider the indefinite admittance matrix $A_aYA_a^T$ with a row and column deleted to form AYA^T. Any QYR^T can be realized by performing rank-preserving row and column operations on AYA^T, that is, $QYR^T = DAYA^TE^T$ where D and E are nonsingular. In an indefinite admittance matrix, every row and column sums to zero, and since the individual values of admittance are left unspecified, this is the only way that row and column operations can produce zero entries. Thus it is only possible to perform a row (column) operation which produces a zero entry in AYA^T if the deleted row (column) has a zero entry in the column (row) in which the zero is produced. Once such a row (column) operation is performed, the column (row) in which the zero entry was produced no longer sums to zero, and so can no longer be used to produce a zero entry. Thus the maximum possible number of zeros which can be produced is equal to the number in the deleted row and column.

Corollary 6. An upper bound on the number of essential zeros of an optimum QR^T for a connected graph is given by twice the number of edges by which the graph fails to be a complete graph, that is, by $n(n-1) - 2e$ where n is the number of vertices of the graph and e is the number of edges.

Proof. Let A_a be the complete incidence matrix for a graph not containing isolated vertices. Then the matrix $A_aA_a^T$ contains two zeros for every non-adjacent vertex pair.

Theorem 7. For any connected graph G, the incidence matrix A with vertex v_i as reference is such that AA^T contains at worst $n-1-\rho_{vi}$ less zero pairs than the maximum attainable number where n is the number of vertices in G and ρ_{vi} is the degree of v_i.

Proof. By Theorem 6 the maximum attainable number of zeros is at most the number of zeros of $A_aA_a^T$. Deleting the row and column corresponding to v_i from $A_aA_a^T$ converts $A_aA_a^T$ into the AA^T matrix described in the theorem. The v_i row and column of $A_aA_a^T$ contain a total of two zeros for each vertex which is not adjacent to v_i, and there are $n-1-\rho_{vi}$ such vertices.

Theorem 8. An optimum QR^T matrix for a connected non-separable graph contains at most 2n - 6 more zeros than any AA^T matrix of the graph, where n is the number of vertices of the graph and A is an incidence matrix.

Proof. The matrix $A_aA_a^T$ for a connected non-separable graph contains at least three nonzero entries in every row and in every column, so if a row and column are deleted to form an AA^T matrix with m zeros, the maximum number of zeros in any QR^T matrix is $m + 2(n-3)$.

Theorem 9. The maximum possible percentage improvement of an optimum QR^T matrix over an optimum AA^T matrix is given by $\frac{2}{n-2} \times 100$ for any connected graph.

Proof. Consider the matrix $A_aA_a^T$. Assume that the i^{th} row has m zeros. If the corresponding vertex is an optimum choice for reference for an AA^T matrix, every other row has at least m zeros, and the AA^T will have at least nm - 2m zeros where n is the number of vertices in the graph. Since any QR^T matrix can have at most nm zeros then, the maximum possible fractional improvement is $\frac{nm - (nm-2m)}{nm - 2m} = \frac{2}{n-2}$.

Theorem 10. Let m_i be the number of vertices in the subgraph G_i composed of the edges of a seg q_i together with their endpoint vertices and let p_i be the number of separate parts of G_i. Then for a given graph a seg q_i is adjacent to at least $m_i - p_i - 1$ other members of any basis set of segs for the graph.

Proof. The number of branches in a forest for the subgraph is $m_i - p_i$ (Busacker and Saaty (1) Theorem 1-6), hence $m_i - p_i$ is the number of adjacent vertex pairs of the subgraph which must be partitioned by segs in any basis. By Theorem 2, q_i can be the only seg partitioning at most one vertex pair, so q_i must be adjacent to at least $m_i - p_i - 1$ segs in any basis.

Corollary 10a. A star q_{vj} is adjacent to at least $\rho_{vj} - 1$ other members of any basis set of segs for a graph, where ρ_{vj} is the degree of the vertex corresponding to q_{vj}.

Proof. By Theorem 10, q_{vj} is adjacent to at least $m_{vj} - p_{vj} - 1$ other segs of any basis. But since q_{vj} is a star $p_{vj} = 1$ and $m_{vj} - 1 = \rho_{vj}$.

Corollary 10b. The orthogonality of a seg q_i is at most $(n-2) - (m_i - p_i - 1) = n - 1 - m_i + p_i$ in any basis set of segs, where n is the number of vertices in the graph. The orthogonality of a star q_{vj} is at most $n - 1 - \rho_{vj}$ in any basis set of segs.

Proof. The corollary is a restatement of Theorem 10 and Corollary 10a in terms of orthogonality rather than adjacency.

Theorem 11. In a complete set of segs Q_a for a graph having n vertices, any given seg q_i is orthogonal to $2^{(n - 1 - m_i + p_i)} - 1$ other members of Q_a.

Proof. Consider the subgraph composed of the edges of q_i together with their m_i endpoint vertices. First assume the subgraph is connected.

Then there will be $n - m_i$ stars which are not adjacent to q_i, and all segs obtainable as a sum of these stars taken two at a time, three at a time, etc., will also be orthogonal to q_i. The sum of a sequence of p objects taken 1, 2, . . ., p at a time is $2^p - 1$, so there will be $2^{(n - m_i)} - 1$ possible segs orthogonal to q_i. If now q_i is assumed to have p_i separate parts, segs composed of combinations of the separate parts along with combinations of nonadjacent stars will also be orthogonal to q_i. Thus there will be $2^{(p_i - 1)}\, 2^{(n - m_i)} - 1$ or $2^{(n - 1 - m_i + p_i)} - 1$ segs of Q_a orthogonal to q_i.

Lemma 12a. In a basis set of segs composed entirely of stars, every star q_{vj} is adjacent to precisely $\rho_{vj} - a_{vj}$ other members of the basis, where a_{vj} is a number equal to 1 if vertex v_j is adjacent to the reference vertex and zero otherwise.

Proof. If the vertex v_j corresponding to q_{vj} is not adjacent to the reference vertex v_n, every edge of q_{vj} will be in precisely two segs, q_{vj} and the seg corresponding to the other vertex incident upon this edge, hence q_{vj} will be adjacent to precisely ρ_{vj} other segs of the basis. If vertex v_j is adjacent to the reference vertex v_n, the edge joining v_j to v_n will be in no other seg but q_{vj}, and all other edges incident upon v_j will be in precisely two segs as before, hence q_{vj} will be adjacent to precisely $\rho_{vj} - 1$ other segs of the basis.

Lemma 12b. If all the segs in a basis set are restricted to be stars, the orthogonality of the set is maximized by choosing a highest-degree vertex as reference.

Proof. By Lemma 12a and Corollary 10a, a basis set of segs composed entirely of stars has the property that for any specified reference vertex each star is adjacent to the fewest possible other stars. Choosing any highest-degree vertex as reference ensures that the remaining stars have minimum ρ and that a maximum number of them are adjacent to the reference vertex.

Theorem 12. If for any given graph there are no multiple segs q_i with the property that $m_i - p_i \leq \rho_{vj} \leq \rho_{vn}$, where v_j and v_n are vertices belonging to w_i and $\overline{w}_i$ respectively, then a set of stars with a highest degree vertex as reference is a maximally orthogonal basis set of segs.

Proof. Theorem 10 and the hypothesis of the theorem ensure that no multiple segs exist for the given graph which can be adjacent to fewer other segs of any basis than a set of stars. Lemmas 12a and 12b then complete the proof.

Theorem 13. If a seg q_i which is not a cut-set is independent of a set of segs F and is adjacent to m of the members of F, there exists another seg q_j which is a cut-set and is also independent of F and is adjacent to at most m of the members of F.

Proof. Let w_i and $\overline{w}_i$ be the vertex partitions generated by seg q_i. Then if q_i is not a cut-set one of the vertex sets, say w_i, must be further divisible into vertex sets w_j and w_k, where the union of w_j and w_k equals w_i and the intersection of w_j and w_k is void, in a manner such that no vertex of w_j is adjacent to any vertex of w_k. Consider the segs q_{wj} and q_{wk} corresponding to vertex sets w_j and w_k. q_i is the sum of q_{wj} and q_{wk}, so at least one of the latter must be independent if q_i is.

Further, q_{wj} and q_{wk} are edge-disjoint, so q_i is adjacent to every seg which is adjacent to either q_{wj} or q_{wk} or both. Thus at least one of q_{wj} and q_{wk}, say q_{wj}, can replace q_i in an independent set of segs with no increase in adjacency. If q_{wj} is a cut-set the theorem is proved; if not, vertex set w_j can be further subdivided and the above process repeated until a cut-set is obtained.

Corollary 13. No maximally orthogonal basis set of segs need contain a seg which is not a cut-set.

Proof. Suppose some maximally orthogonal basis set of segs contains segs which are not cut-sets. Then by Theorem 13, each of these segs can be replaced by another which is a cut-set with no increase in adjacency. The set of cut-sets so formed must then also be a maximally orthogonal basis set.

Theorem 14. The n-1 independent node variables associated with a set of generalized nodal equations can be node-pair voltages if and only if the set of segs associated with the node transformation is a fundamental set, that is, if and only if every seg in the set is the only seg partitioning some vertex pair of the graph.

Proof. Assume that the graph is fully connected, that is, every vertex pair is connected by an edge. No generality is lost by this assumption since any graph may be augmented by zero admittance-weight edges without affecting the nodal equations associated with the graph. The node transformation is then $Q^T V_n = V_b$ where Q^T is the transpose of a seg matrix, V_n is a column matrix whose entries are node variables, and V_b is a column matrix whose entries are node-pair voltages. V_b contains a row for each possible node-pair voltage of the graph and V_n contains one less row than the network under consideration has nodes, which is the largest number of voltages that can be independently specified for an electrical network. The node transformation is a singular transformation expressing a set of variables, the entries of V_b, in terms of a basis set, the entries of V_n.

First suppose that Q is a fundamental seg matrix. Then every row of Q will contain an entry not in any other row, hence Q must contain a unit matrix. After a suitable reordering of the rows of Q^T and V_b, the node transformation may be written in partitioned form as

$$\begin{bmatrix} U \\ \hline Q_{12}^T \end{bmatrix} \begin{bmatrix} V_n \end{bmatrix} = \begin{bmatrix} V_{b\ 11} \\ \hline V_{b\ 12} \end{bmatrix}$$

from which $UV_n = V_n = V_{b\ 11}$. Since every entry in V_b is a node-pair voltage, every entry in V_n must also be a node-pair voltage.

Conversely, suppose the entries of V_n are all node-pair voltages. Then this set of node-pair voltages must be a subset of those node-pair voltages which are entries of V_b. Thus, after a suitable reordering of the rows of V_b, V_b may be partitioned in such a fashion that it contains V_n as a submatrix. After conformally partitioning Q^T, the node transformation equation becomes

$$\begin{bmatrix} Q_{11}^T \\ \hline Q_{12}^T \end{bmatrix} \begin{bmatrix} V_n \end{bmatrix} = \begin{bmatrix} V_n \\ \hline V_{b\ 12} \end{bmatrix} .$$

Expanding, $Q_{11}^T\ V_n = V_n = UV_n$, and $(Q_{11}^T - U)\ V_n = 0$. If Q_{11}^T is not a unit matrix, then a linear combination of the elements of V_n will be equal to zero, contradicting the hypothesis that a set of node variables is an independent set. Therefore, Q_{11}^T is a unit matrix. Thus, Q contains a unit matrix of maximum possible rank and so must be a fundamental seg matrix.

Theorem 15. A basis set of segs for a graph will be a fundamental set if and only if the set does not contain any interlocked segs.

Proof. First assume that the basis set of segs contains an interlocked pair of segs, q_i and q_j. Then this pair of segs divides the vertices of the graph into four nonvoid sets, w_1, w_2, w_3, and w_4. Let q_i be the seg partitioning vertex sets w_1 and w_2 from w_3 and w_4 and let q_j be the seg partitioning w_1 and w_4 from w_2 and w_3. If q_i is the only seg partitioning a vertex pair, the vertices of the pair must belong to sets w_1 and w_4 or to w_2 and w_3, and if q_j is the only seg partitioning a vertex pair, the vertices of this pair must belong to sets w_1 and w_2 or to w_3 and w_4. Any combination of these possibilities involves three of the four vertex sets, say w_1, w_2, and w_3. By Theorem 2, every vertex pair in which one vertex is in w_4 and the other in the union of w_1, w_2, and w_3 must be partitioned by at least two segs. But if the set of segs is to be fundamental, there must be n-1 vertex pairs (for a graph with n vertices) each partitioned by only one seg, and the independence of the set requires that one of these vertex pairs have one vertex in w_4 and the other in the union of w_1, w_2, and w_3. Thus a basis set of segs containing interlocked segs cannot be a fundamental set.

Conversely, assume that the basis set of segs is not a fundamental set. Then at least one seg of the set, say q_i, must be such that every vertex pair partitioned by q_i is also partitioned by another seg of the set. If we designate any vertex of the graph as reference and characterize each seg q_i by the vertex set w_j it generates which does not contain the reference vertex, then if w_i is not a proper subset of some w_j every vertex in w_i must be contained in another vertex set. If none of these vertex sets correspond to interlocked segs, the union of some vertex-disjoint subset of them will be w_i, thus the segs cannot be an independent set. If every vertex in w_i is not contained in another vertex set, the requirement that every vertex pair partitioned by q_i be also partitioned by other segs requires that w_i be a proper subset of some w_j and that every vertex that is in w_j but not w_i must be contained in another vertex set. The result is as in the previous case. Thus a non-fundamental basis set of segs must contain interlocked segs.

Corollary 15. The node variables associated with a set of generalized nodal equations can be node-pair voltages if and only if the set does not contain any interlocked segs.

Proof. Apply Theorem 15 to Theorem 14.

Theorem 16. Let a seg q_i be a member of a seg basis for a graph G. Then if q_i is not interlocked with any other member of the basis, q_i must be adjacent to at least $m_i - b_i$ other members of the basis, where m_i is the number of vertices which are adjacent to an edge of q_i and b_i is a number such that:

$b_i = 2$ if q_i is the only seg separating a vertex pair and both vertices of the pair are adjacent to an edge of q_i

$b_i = 1$ if q_i is the only seg separating a vertex pair and precisely one vertex of the pair is adjacent to an edge of q_i

$b_i = 0$ otherwise.

Proof. Consider the subgraph G' composed of the edges belonging to

q_i and their endpoint vertices. Since the set of segs is independent, every vertex of G' must be partitioned from every other vertex of G by at least one seg, and no seg can be the only seg partitioning more than one vertex pair. Since by hypothesis none of the segs of the basis are interlocked with q_i, q_i will be adjacent to at least one seg for each vertex of G' except those one or two vertices which are part of a vertex pair separated only by q_i.

Theorem 17. Let Q be a fundamental seg matrix and let D be the nonsingular transformation matrix relating Q to an incidence matrix A, so that Q = DA, and let a consistent reference convention be adopted for Q and A. Then for a suitable ordering of the rows and columns of Q and A, D can always be written as the sum of a unit matrix and a strictly lower triangular matrix containing only zeros and ones as entries. Moreover, D can be partitioned into a direct sum matrix in which each of the diagonal submatrices D_i has a last row in which all the entries are ones. Deleting the last row and column from D_i produces a new submatrix in which either the last row has all its entries ones or the submatrix can be partitioned into a direct sum matrix in which each of the submatrices so determined has a last row in which all the entries are ones. In either case, the process can be repeated until all such submatrices are reduced to first order matrices.

Proof. The reference vertex, that is, the vertex not corresponding to a star of A, must be a member of at least one vertex pair partitioned by only one seg. By Theorem 15, such segs cannot be interlocked since they are to belong to a fundamental set, hence the segs are vertex disjoint and can be written as sums of disjoint sets of stars. Each such seg is the only seg partitioning a vertex pair, so one of the stars which sum to the seg cannot be in any other seg. The vertex for this star may then be considered the reference vertex for the subgraph corresponding to the set of stars, and the argument repeated for each subgraph, sub-subgraph, etc.

Corollary 17. Let Y_{ij} be an element of the matrix QYQ^T for a network with a graph G, and let Q be a fundamental seg matrix. Then with a consistent reference convention and suitable ordering of the vertices of G,

$$Y_{ij} = \sum_{p=a\leq i}^{i} \sum_{q=b\leq j}^{j} y_{pq}$$

where y_{pq} is an indefinite admittance matrix element for the network corresponding to G.

Proof. The corollary is a direct consequence of Theorem 17 with the matrix operations written as summations and the fact that y_{pq} is an element of AYA^T for G.

Theorem 18. Divide the vertices of any given graph into four all-inclusive mutually-exclusive nonvoid sets, w_1, w_2, w_3, and w_4. Designate the seg corresponding to vertex set w_k as q_{wk}. Define a seg q_i to be that seg whose corresponding vertex set w_i is the union of w_1 and w_2 and define a seg q_j to be that seg whose corresponding vertex set w_j is the union of w_2 and w_3. Let m_k and b_k be as defined in Theorem 16 for a seg q_k in any specified basis not containing interlocking segs. Then if q_i can replace q_{w1} in an otherwise fixed basis containing q_{w3} and if $(m_{w1} - b_{w1}) - (m_i - b_i) = c$, where c is a number, it is true that $(m_{w3} - b_{w3}) - (m_j - b_j) \leq - c$.

Proof. Let $(m_{w1} - b_{w1}) - (m_i - b_i) = c$ and let $(m_{w3} - b_{w3}) - (m_j - b_j) = d$. Adding these two equations together results in

$$c + d = (m_{w1} - b_{w1}) - (m_i - b_i) + (m_{w3} - b_{w3}) - (m_j - b_j)$$

Define the quantity $n_{k,p\ldots q}$ to be the number of vertices in vertex set w_k which are adjacent to vertices in vertex set w_p and ... and w_q. Then the m_k terms may be expanded as follows:

$$\begin{aligned}
m_{w1} &= (n_{1,2} + n_{1,3} + n_{1,4} - n_{1,23} - n_{1,24} - n_{1,34} + n_{1,234}) \\
&\quad + n_{2,1} + n_{3,1} + n_{4,1} \\
m_{w3} &= (n_{3,1} + n_{3,2} + n_{3,4} - n_{3,12} - n_{3,14} - n_{3,24} + n_{3,124}) \\
&\quad + n_{1,3} + n_{2,3} + n_{4,3} \\
m_i &= (n_{1,3} + n_{1,4} - n_{1,34}) + (n_{2,3} + n_{2,4} - n_{2,34}) \\
&\quad + (n_{3,1} + n_{3,2} - n_{3,12}) + (n_{4,1} + n_{4,2} - n_{4,12}) \\
m_j &= (n_{1,2} + n_{1,3} - n_{1,23}) + (n_{2,1} + n_{2,4} - n_{2,14}) \\
&\quad + (n_{3,1} + n_{3,4} - n_{3,14}) + (n_{4,2} + n_{4,3} - n_{4,23})
\end{aligned}$$

Then
$$\begin{aligned}
-(c + d) &= (n_{2,4} - n_{2,14}) + (n_{2,4} - n_{2,34}) + (n_{4,2} - n_{4,12}) \\
&\quad + (n_{4,2} - n_{4,23}) + (n_{1,24} - n_{1,234}) + (n_{3,24} - n_{3,124}) \\
&\quad + (b_{w1} - b_i) + (b_{w3} - b_j)
\end{aligned}$$

Now investigate the b_k terms to find conditions for which b_i can be greater than b_{w1}. By hypothesis q_i and q_{w1} can be interchanged in the basis under consideration. Thus, if q_i is the only seg partitioning a given vertex pair in the basis, then q_{w1} will also be the only seg partitioning this vertex pair if it replaces q_i. This requires that one vertex of the pair be in vertex set w_1 and the other in w_4. If the vertex in w_1 is incident upon an edge of q_i it must also be incident upon an edge of q_{w1}. The vertex in w_4 can be incident upon an edge of q_i but not q_{w1} only if the vertex is adjacent to a vertex in w_2 but not w_1. Thus $(n_{4,2} - n_{4,12} - b_i + b_{w1}) \geq 0$. By symmetry, $(n_{4,2} - n_{4,23} - b_j + b_{w3}) \geq 0$ also, so

$$-(c + d) = (n_{2,4} - n_{2,14}) + (n_{2,4} - n_{2,34}) + (n_{4,2} - n_{4,12} - b_i + b_{w1}) + (n_{4,2} - n_{4,23} - b_j + b_{w3}) + (n_{1,24} - n_{1,234}) + (n_{3,24} - n_{3,124}).$$

Now all the bracketed terms on the right hand side of this equation are equal to or greater than zero, so $-(c + d) \geq 0$ and $d \leq -c$.

Lemma 19. Let interlocking segs q_i and q_j be members of a basis set of segs P for a graph, and let q_i and q_j partition the vertices of the graph into four sets, w_1, w_2, w_3, and w_4, in a manner that the union of w_1 and w_2 is a vertex set corresponding to q_i and the union of w_2 and w_3 is a vertex set corresponding to q_j. Define the set of segs P' to be the basis set P with q_i and q_j deleted, and let q_{wk} be a seg partitioning vertex set w_k from the remaining vertices of the graph. Then at least one of the sets (P', q_{w1}, q_{w3}) and (P', q_{w2}, q_{w4}) will also be a basis.

Proof. Apply Corollary 4b to every multiple seg of P'. Then all but three of the vertices of the graph will correspond to stars. Further, by Theorem 1 no two of these three vertices can be in the same w_k since the segs are independent, so either w_1 and w_3 each contain one vertex or w_2 and w_4 each contain one vertex. The reduction of the members of P' to stars is unaffected if q_i and q_j are replaced by other segs partitioning the same three vertices from one another, hence at least one of (P', q_{w1}, q_{w3}) and (P', q_{w2}, q_{w4}) will be a basis.

Theorem 19. If a basis set of segs for a graph contains an interlocking pair of segs in which both segs of the pair are not further interlocked with any other segs of the set, then the interlocking pair may be replaced by a non-interlocking pair with no increase in adjacency.

Proof. Let segs q_i and q_j be an interlocking pair as defined in the Theorem, and let w_1, w_2, w_3, w_4 q_{wk}, P, and P' be as defined in Lemma 19. Then by Lemma 19, q_i and q_j can be replaced in the basis by either q_{w1} and q_{w3} or by q_{w2} and q_{w4}. No generality is lost by assuming that the former case is true because the latter case can be converted into the former by a suitable relabelling of the vertex sets. Let s_{mn} represent the set of edges connecting vertex sets w_m and w_n, and let $\sigma(ij, kl, \ldots, mn)$ be the number of segs belonging to P' which contain edges belonging to sets s_{ij}, s_{kl}, ..., and s_{mn} but none of the other sets s_{pq}. Let $\lambda_{qk}(P')$ be the number of segs of P' adjacent to seg q_k, and let $\lambda(P_j)$ represent the adjacency of a particular basis P_j. Then the number of segs of P' which are adjacent to q_i less the number adjacent to q_{w1} may be written as given below.

$$\begin{aligned}\lambda_{qi}(P') - \lambda_{qw1}(P') = {} & \sigma(23) + \sigma(24) + \sigma(23,24) \\ & + \sigma(23,34) + \sigma(24,34) \\ & + \sigma(23,24,34) - \sigma(12) - \sigma(12,34) .\end{aligned}$$

Similarly, the number of segs of P' which are adjacent to q_j less the number adjacent to q_{w3} is

$$\lambda_{qj}(P') - \lambda_{qw3}(P') = \sigma(12) + \sigma(24) + \sigma(12,14) + \sigma(12,24) + \sigma(14,24) + \sigma(12,14,24) - \sigma(23) - \sigma(14,23)$$

Now if s_{13} is non-void, q_{w1} will be adjacent to q_{w3} and q_i will be adjacent to q_j. If s_{13} is void, q_{w1} will not be adjacent to q_{w3}, but q_i may or may not be adjacent to q_j. Thus in any event the adjacency of basis $P_1 = (P', q_i, q_j)$ less the adjacency of basis $P_2 = (P', q_{w1}, q_{w3})$ is equal to or greater than the sum of the two previously derived equations.

$$\lambda(P_1) - \lambda(P_2) \geq 2\sigma(24) + \sigma(12,14) + \sigma(12,24) + \sigma(14,24) + \sigma(23,24) + \sigma(23,34) + \sigma(24,34) + \sigma(23,24,34) + \sigma(12,14,24) - \sigma(12,34) - \sigma(14,23)$$

Those segs which contain edges of s_{12} and s_{34} or edges of s_{14} and s_{23} must necessarily be interlocking with q_i or q_j or both, and by hypotheses P' contains no such segs. The remaining terms are all inherently non-negative, so $\lambda(P_1) - \lambda(P_2) \geq 0$, completing the proof.

Corollary 19. If a maximally orthogonal basis set of segs for a graph contains interlocking segs in a manner that both segs of every interlocked pair are not further interlocked with any other members of the basis, then a fundamental set of segs exists which is also a maximally orthogonal basis.

Proof. Apply Theorem 19 to every such interlocking pair until none remain. Then the resulting set is at least as orthogonal as the original set and is, by Theorem 15, a fundamental set.

III. DISCUSSION

As a first step in this investigation, some properties of basis sets of segs were established in Theorems 1 through 4 and their corollaries. Theorems 1 and 2 both have obvious algebraic analogs. Because of the very simple structure of incidence matrices, Theorem 3 has practical value as a means of constructing more complicated seg matrices or for determining if a given set of segs is independent. Theorem 4 and its corollaries are essentially replacement theorems concerned with the existence of segs capable of replacing a given seg in an independent set.

The next several theorems deal with bounds on the number of essential zeros attainable as nodal equation coefficient matrix entries. It was established in the Introduction that this number is the same as the number of essential zeros of QR^T where Q and R are seg matrices. In all subsequent discussion the term "optimum QR^T" will be understood to refer to the nodal equation coefficient matrix having the maximum possible number of essential zero entries, and the term "optimum QQ^T" will refer to a similar matrix with the additional restriction that the constituent seg matrices are equal. Theorem 5 gives a sufficient condition for which the matrix obtained by deleting a row and corresponding column from the indefinite admittance matrix $A_aYA_a^T$ is an optimum QYR^T matrix. Theorem 6 gives the very basic result that the number of zero entries in an indefinite admittance matrix is an upper bound on the number attainable in an optimum QR^T matrix. It then follows that deleting a row and corresponding column from the indefinite admittance matrix to form the matrix AYA^T loses just those zeros which were in the row and column deleted, and Theorems 7, 8, and 9 are concerned with the number or percentage improvement in the number of zero entries in an optimum QYR^T matrix over the number in an AYA^T matrix.

The fact that an optimum QYR^T matrix regains as many as possible of the zeros lost by deleting a row and column from $A_aYA_a^T$ suggests the interpretation that the search for this Q and R is equivalent to performing row and column operations on AYA^T to transfer zeros from the deleted row and column by means of the indefinite admittance matrix property thatall rows and columns sum to zero. If for example the deleted row has a zero entry in the k^{th} column, the sum of all rows having nonzero entries in the k^{th} column produces a zero in this position. If this sum has more off-diagonal zero entries than one of its constituent rows, it can replace that row and thus in effect transfer a zero out of the deleted row.

The next topic investigated will be the determination of the minimum number of segs of a basis to which a given seg can be adjacent. Since any two segs are adjacent if they have a common edge, one might expect a close connection between the number of edges of a seg and the minimum number of segs to which it is adjacent. The only complication which arises is that a seg can contain an edge which does not couple it to any other segs to which it is not already adjacent. Such edges have the geometric appearance of a cross-coupling. For example, in Figure 1 edge ad will cause seg q_i to be adjacent to any other seg partitioning vertices a and d. But if edge ad is removed from the graph, any seg partitioning vertices a and d will still be adjacent to q_i via one or more of edges ac, bc, and bd. One equivalent way to count the edges of a seg which are not cross-coupled is to count the number of vertices which are incident

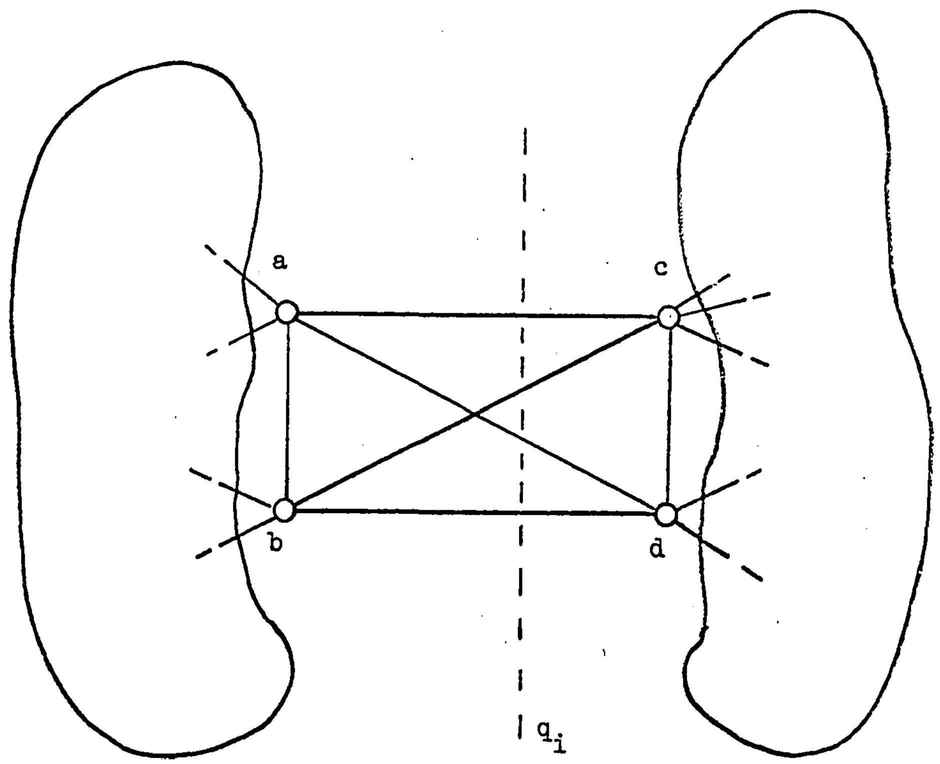


Figure 1. An example of a seg containing a cross-coupled edge

upon an edge of the seg and subtract the number of separate parts of the subgraph consisting of the edges of the seg together with their endpoint vertices. Thus Theorem 10 establishes that in any basis a seg q_i must be adjacent to at least as many segs as one less than the number of non-cross-coupled edges of q_i. If q_i is a star the statement of Theorem 10 may be simplified because the quantity $m_i - p_i$ then reduces to simply the degree of the vertex corresponding to the star. Corollary 10a is a re-statement of Theorem 10 for this condition. Corollary 10b restates Theorem 10 and Corollary 10a in terms of orthogonality rather than adjacency.

Interestingly enough, it is a considerably easier task to determine how many of the $2^{n-1} - 1$ segs of a complete set of segs for a graph are orthogonal to a given seg than to choose an independent set of n-1 segs of maximum orthogonality. It is proven in Theorem 11 that a seg q_i belonging to a complete set of segs Q_a is orthogonal to $2^{(n-1-m_i-p_i)} - 1$ members of Q_a. Arranging all the segs of Q_a in order of orthogonality and choosing the first n-1 independent segs as a basis Q will often but not always result in a maximally orthogonal basis since a seg q_i can be orthogonal to more members of Q_a but less members of Q than a seg q_j. A notable example of this type of seg is one which achieves a low value of $m_i - p_i$ through large p_i rather than small m_i. For example, q_k in Figure 2 is such that $m_k - p_k = 6 - 3 = 3$. Every other possible seg q_i for this graph has $m_i - p_i \geq 3$, hence q_k is orthogonal to a maximum number of other members of Q_a, yet q_k does not belong to any maximally orthogonal basis. The reason is apparent from Figure 2. Many of the segs orthogonal to q_k are those in which, for example, vertices b and c or f and g are in the same

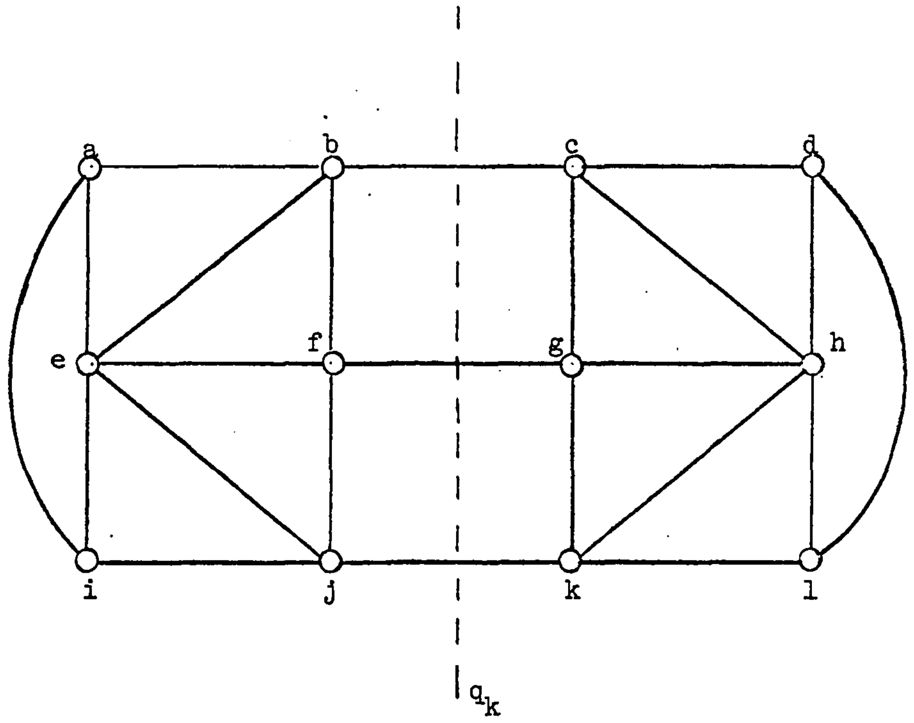


Figure 2. Example of a seg which is orthogonal to a maximum number of the members of Q_a but is not a member of any maximally orthogonal basis

vertex set, and such segs will be orthogonal to relatively few other segs. In the special case that for every multiple seg q_i of a graph the number $m_i - p_i$ is greater than the degree of every vertex in one of the two vertex sets generated by that multiple seg, it is true that one choice of the first n-1 independent segs of a complete set Q_a arranged in order of orthogonality will be a set of stars. It is shown in Theorem 12 that such a set of stars will be a maximally orthogonal set. The hypothesis of Theorem 12 is a sufficient but not necessary condition for a set of stars to be a maximally orthogonal basis, as evidenced by graphs such as the one shown in Figure 2.

One general statement which can be made is that the segs comprising a maximally orthogonal set can always be chosen to be cut-sets, which is proven in Corollary 13. If to the hypothesis of Theorem 13 is added the restriction that seg q_i not be interlocked with any member of F, the corollary would be changed to state that maximally orthogonal bases not containing interlocking segs cannot contain segs which are not cut-sets.

By Theorems 14 and 15 and Corollary 15, the following three statements are completely equivalent:

1. The node variables associated with a basis set of segs are all node-pair voltages.
2. The basis set of segs is a fundamental set.
3. The basis set of segs contains no interlocking segs.

Theorem 16 then gives the minimum number of segs of a basis which can be adjacent to a given seg with any of the above restrictions added and for either of two preconditions or for no preconditions on the remaining members of a basis. Note that if the basis is restricted to consist of non-interlocked segs only the number of segs to which a given seg must be adjacent is always greater than the number which can be achieved with the ban on interlocked segs removed if the subgraph corresponding to the given seg has more than one separate part. This verifies that interlocked segs are necessary to achieve a minimum adjacency for a seg whose corresponding subgraph has more than one separate part. As discussed before, however, achieving a minimum adjacency for any one seg by no means ensures that the set so formed will have maximum orthogonality.

The next topic to be discussed is the determination of an optimum QQ^T if Q is restricted to be a fundamental seg matrix. Finding an optimum QQ^T is equivalent to selecting a set Q of n-1 independent segs from a complete set of segs Q_a in a manner that the orthogonality of Q is at least as great as that of any other set of n-1 independent segs of Q_a. A fundamental set of segs can be considered to be based on a tree of a complete graph, and every graph can be converted into a complete graph without affecting the nodal equations describing it by augmenting it with zero admittance-weight edges. It would be highly desirable if an algorithm could be found which would lead to a tree corresponding to a maximally orthogonal fundamental set of segs, but in all probability such an algorithm does not exist. Trees corresponding to maximally orthogonal fundamental sets of segs for different graphs have little in common, and the addition or deletion of a single edge from a graph can profoundly change the character of such a tree. This large sensitivity of the choice of an optimum set of segs to the edge structure was found to be typical of those cases for which an incidence set was not optimum. As an extreme if somewhat pathological example, consider the graph and set of segs given in Figure 3. The indicated set of segs gives the best possible QQ^T matrix--every off-diagonal entry is zero. If an edge is added between vertices a and f, the same set of segs now gives the worst possible QQ^T matrix, one with no zero entries at all.

The most promising procedure for finding an optimum choice of fundamental segs appears to be one which works directly with the matrix product QQ^T. As a first step Theorem 17 gives a characterization of a fundamental seg matrix Q in terms of a specific form for the transformation matrix which operates on an incidence matrix to produce Q. This in turn leads to Corollary 17, which states that if Q is specified to be a fundamental seg matrix for a graph G, then an element Y_{ij} of the matrix QYQ^T may be expressed as

$$Y_{ij} = \sum_{p=a\leq i}^{i} \sum_{q=b\leq j}^{j} y_{pq} ,$$

where y_{pq} is an indefinite admittance matrix element for the electrical network corresponding to G. The form of this general term indicates that all diagonal elements of QYQ^T must be nonzero if G is connected. This is best seen by writing the diagonal term in expanded form as

$$\begin{aligned} Y_{ii} = (y_{aa} + y_{a(a+1)} + \cdots + y_{ai}) + (y_{(a+1)a} + y_{(a+1)(a+1)} \\ + \cdots + y_{(a+1)i}) + \cdots\cdot + (y_{ia} + y_{i(a+1)} + \cdots + y_{ii}) \ . \end{aligned}$$

Now Y_{ii} is zero only if each bracketed term is zero. But the k^{th} bracketed term is zero only if the k^{th} vertex is adjacent to a subset of the a^{th} through i^{th} vertices and no others, and this cannot be true for all

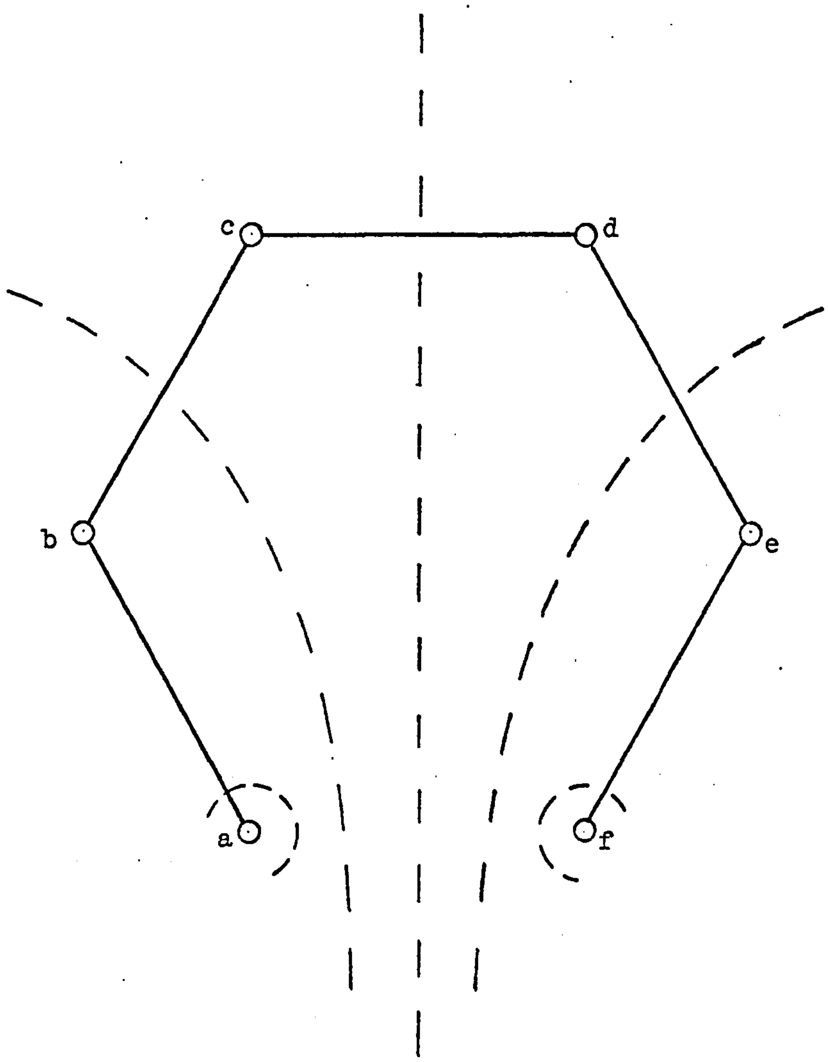


Figure 3. Example of sensitivity of orthogonality of a set of segs to the edge structure of a graph

bracketed terms without contradicting the hypothesis that the graph is connected.

The form of an element of QYQ^T also allows the following conclusions to be drawn concerning the nature of the row and column operations forming QYQ^T from AYA^T:

1. All necessary operations produce zero entries.
2. The utility of any given operation can be affected by a subsequent operation.
3. Every necessary row (column) operation involves a summing of rows (columns) in which every constituent row (column) contributes to the production or maintenance of a zero entry.

Examination of the general term reveals that if any entry is zero, then that zero would also have been produced by the responsible row or column operation alone. The other operation applied to the term, if any, can then prevent the gaining of a zero but it cannot aid in the zero production process. Interactions between operations are thus wholly negative. This conclusion represents a major difference between fundamental and non-fundamental sets of segs, for if interlocking segs are permitted it is possible that under rather stringent conditions a QYQ^T element can be zero even though neither of the operations affecting the element can by themselves produce a zero. The fact that the number of zeros gained or lost by a particular operation can be affected by a subsequent operation suggests that a procedure leading monotonically to an optimum QYQ^T is not to be found. It is even possible, and will later be demonstrated by an example, that a sequence of row and column operations can result in a net gain of zeros in cases where none of the operations by themselves produce more zeros than they lose. This behavior is possible because two or more operations can lose zeros in the same matrix address. Because of the property that all interactions of operations are negative, only those operations need be considered which produce zeros in off-diagonal positions, and in each such operation the only rows or columns which need be involved are those which are active participants in producing a zero or preventing the loss of a zero.

Two procedures for finding optimum or near-optimum QYQ^T matrices will next be presented and each will be illustrated by means of an example. The first procedure is relatively straightforward and direct and realizes the maximum possible number of zeros in most but not all cases. The second procedure involves substantially more trial and error but is guaranteed to result in an optimum QYQ^T matrix.

The first procedure, hereafter referred to as procedure I, is as follows:

1. Write the $A_aA_a^T$ matrix for the network.
2. Select a row with a minimum number of zeros, say the k^{th}, and cross it and the corresponding column out. If this row has no zeros, the resulting incidence matrix is optimum. If it does, continue the procedure.
3. Choose a column, say the m^{th}, which has a zero in the k^{th} row. Sum the rows which have nonzero entries in the m^{th} column and note whether or not the sum has more off-diagonal zero entries than a constituent row. If not, repeat for all remaining k^{th} row zeros, and continue with row operations producing two k^{th} row zeros at a

time, then three, and so forth until either a row operation is found which gains zeros or all possibilities have been exhausted. If the latter, the procedure is finished; if the former, continue the procedure.

4. The set of rows determined in step 3 will appear as a group if at all in any subsequent operations involving other rows of the matrix. If the row operation determined in step 3 does not lose one or more zeros in any position in the row, perform the row and corresponding column operations. If it does, check to determine if a larger grouping of rows yields a larger net improvement. If not, perform the previously indicated row and column operations. If so, find a set of rows which either loses no zeros or which has as large a net improvement as possible and perform the corresponding row and column operations.

5. Recycle to step 3 and repeat for all combinations of zero-producing operations for which the rows are proper subsets of the operation performed in step 4.

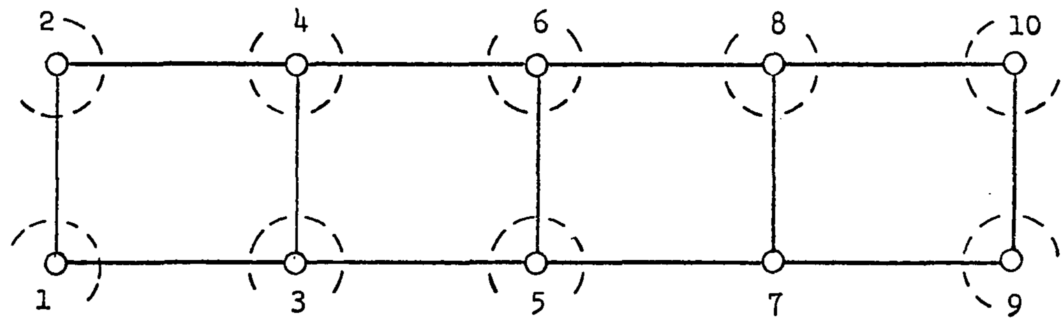
6. Recycle to step 3 and repeat with the restriction that any new row operation will not include any rows used in previous operations unless it includes all other rows of that previous operation.

The $A_a A_a^T$ matrix desired in step 1 is simply an indefinite admittance matrix with all admittances assumed to have unity value, and so is easily written by inspection. In step 2, it doesn't really matter which row is chosen as reference, for if a row without a minimum number of zeros is chosen, the sum of all remaining rows will be found to correspond to a row operation which will be performed in step 3, and replacing a row and column by the sum of all other rows and columns is simply equivalent to selecting a new reference. Theorem 18 justifies not testing combinations containing proper subsets of sets of rows whose sum gives an improvement in the number of zeros in a row. In step 5 it is not necessary to actually test every combination. The only operations which need to be rechecked are those which had previously lost precisely as many zeros as they gained.

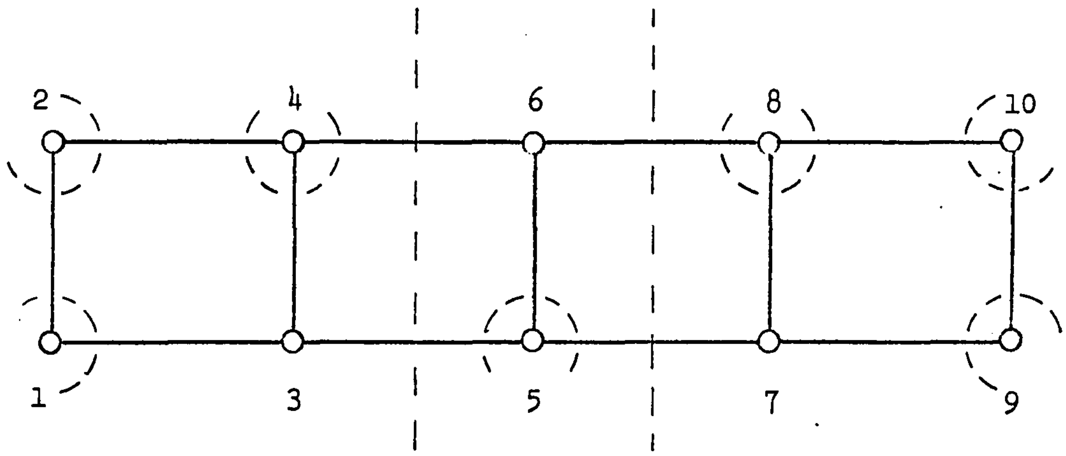
As an example of the application of the procedure, a maximally orthogonal basis set of fundamental segs will be found for the graph of Figure 4. The $A_aA_a^T$ matrix with the row and column corresponding to vertex 7 chosen as reference is given below.

$$A_aA_a^T = \left[\begin{array}{cccccc|c|ccc}
2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 3 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 3 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 3 & 0 & -1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -1 & 0 & 3 & -1 & -1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & -1 & -1 & 3 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2
\end{array}\right]$$

As may be verified from the graph or the above matrix, $A_aA_a^T$ has $n(n-1) - 2e = (10)(9) - (2)(13) = 64$ zero entries, and an optimum choice of reference vertex yields an AA^T matrix with $64 - 2(n-1-\rho_{v7}) = 64 - 2(10-1-3) = 52$ zero entries. There are no row operations producing a



(a)



(b)

Figure 4. Example of a graph showing (a) an optimum set of stars and (b) an optimum choice of a fundamental set of segs

single reference zero in AA^T which result in a net improvement in the number of zeros, nor are there any involving two reference zeros which result in a net improvement. The first row operation found which yields a net improvement is adding rows 1, 2, 3, and 5 to row 4. This operation produces zeros in columns 2 and 3 and loses a zero in column 5 for a net gain of one. Since the operation loses a zero, larger groupings of rows containing rows 1, 2, 3, 4, and 5 are investigated and it is found that adding rows 1, 2, 3, 4, and 5 to row 6 produces one zero without losing any. This row and column operation is then performed, resulting in the matrix below.

$$Q_1 Q_1^T = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

All combinations of the first five rows are then rechecked and it is found that adding rows 1, 2, and 4 to row 3 now produces one zero without losing any. This row operation and the corresponding column operation are then performed.

$$Q_2Q_2^T = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

There remain no row operations involving rows 1, 2, and 3 or the still unused rows 8, 9, and 10 which result in an increased number of zeros, consequently the procedure is finished. Two zero pairs have been gained, so an optimum QQ^T contains 56 zeros for this graph. The resulting optimum fundamental set of segs is shown in Figure 4(b).

In the example just concluded, procedure I led to an optimum set of fundamental segs. It cannot be guaranteed to do so however. To be absolutely certain that a set of fundamental segs is maximally orthogonal, it is necessary to essentially check all combinations of zero-producing operations, not just those which at some stage can in themselves increase the number of zeros in the admittance matrix. While the number of fundamental seg sets containing combinations of segs corresponding to zero-producing operations and stars is likely to be considerably smaller than n^{n-2}, the total number of fundamental sets of segs for a graph with n vertices, it is likely to remain a large number. One further simplification which can reduce the total labor is to make explicit all possible interactions. This will be referred to a procedure II.

Procedure II will be given by example only, using the graph of Figure 5. This graph is an example of one for which procedure I will fail to yield a maximally orthogonal set of fundamental segs. The first step in procedure II is to construct an AA^T matrix with a vertex of maximum degree as reference. For the graph of Figure 5, the only possible reference choice is vertex 10. The only information needed at present is the presence or absence of edges, hence it is convenient to utilize the diagonal matrix addresses as row and column number markers and to mark nonzero entries with the symbol X. Zero entries are simply left blank. The reference row is shown immediately below the AA^T matrix.

$$
\begin{array}{cc}
AA^T = & \begin{bmatrix}
1 & X & X & & X & & & & \\
X & 2 & X & & & & & & \\
X & X & 3 & X & & & & & \\
 & & X & 4 & X & X & & & X \\
 & & & X & 5 & X & & & \\
X & & & X & X & 6 & X & & \\
 & & & & & X & 7 & X & X \\
 & & & & & & X & 8 & X \\
 & & & X & & & X & X & 9
\end{bmatrix} \\
 & \begin{matrix} X & 0 & X & X & 0 & X & X & 0 & X \end{matrix}
\end{array}
$$

There are three zeros in the deleted row, and since the rows which can produce them are disjoint, there are no combinations of operations which need be considered. Label the three zero-producing operations as a, b, and c and mark in the zero entries of the matrix the labels of the

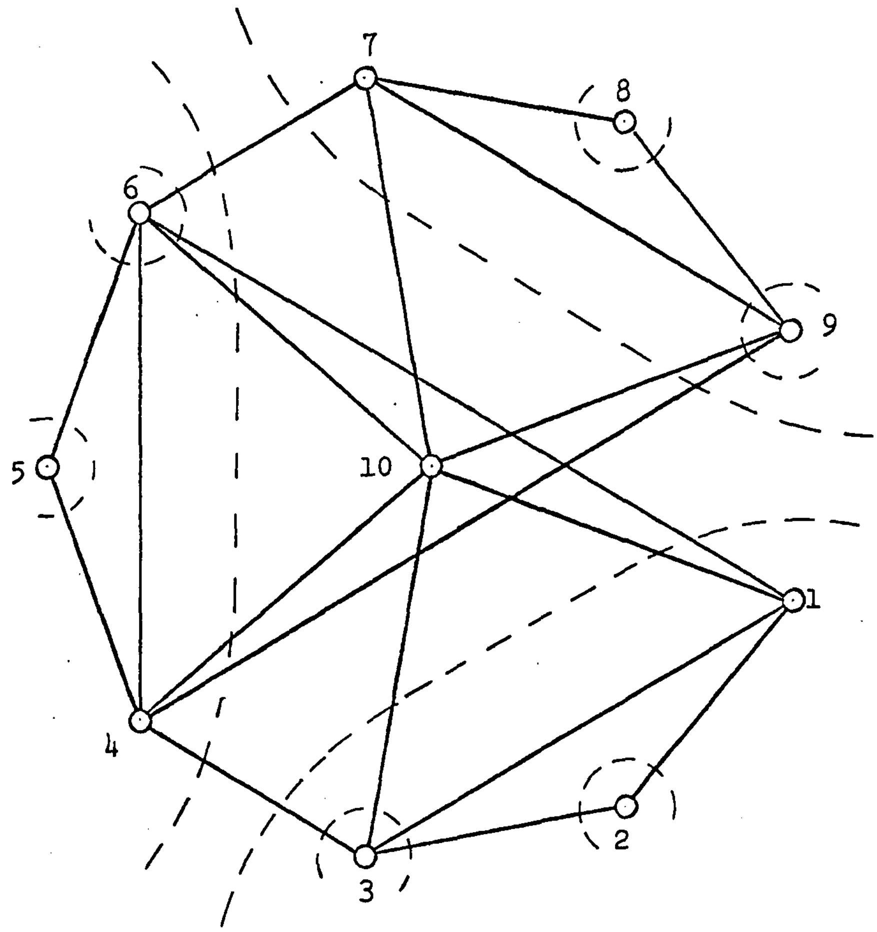


Figure 5. Example of a graph showing an optimum choice of a fundamental set of segs

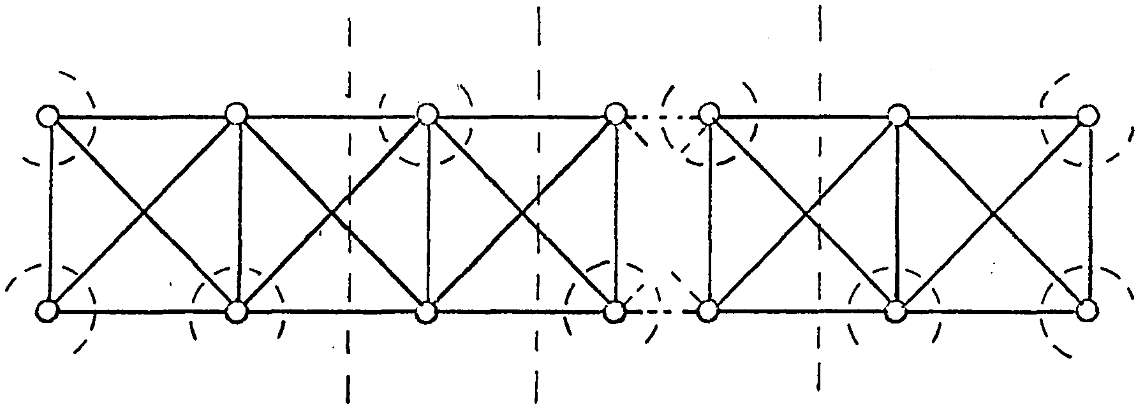
operations which can lose that zero. For example, summing the first three rows to produce the first reference zero can lose zeros in any of the 14, 24, 26, or 36 positions, so these entries and their main diagonal reflections are labelled with the letter corresponding to this operation. When this is done for all three operations, the result is as shown below.

$$\begin{bmatrix} 1 & X & X & ab & b & X & & & \\ X & 2 & X & a & & a & & & \\ X & X & 3 & X & b & ab & & & \\ ab & a & X & 4 & X & X & bc & c & X \\ b & & b & X & 5 & X & b & & b \\ X & a & ab & X & X & 6 & X & c & bc \\ & & & bc & b & X & 7 & X & X \\ & & & c & & c & X & 8 & X \\ & & & X & b & bc & X & X & 9 \end{bmatrix}$$

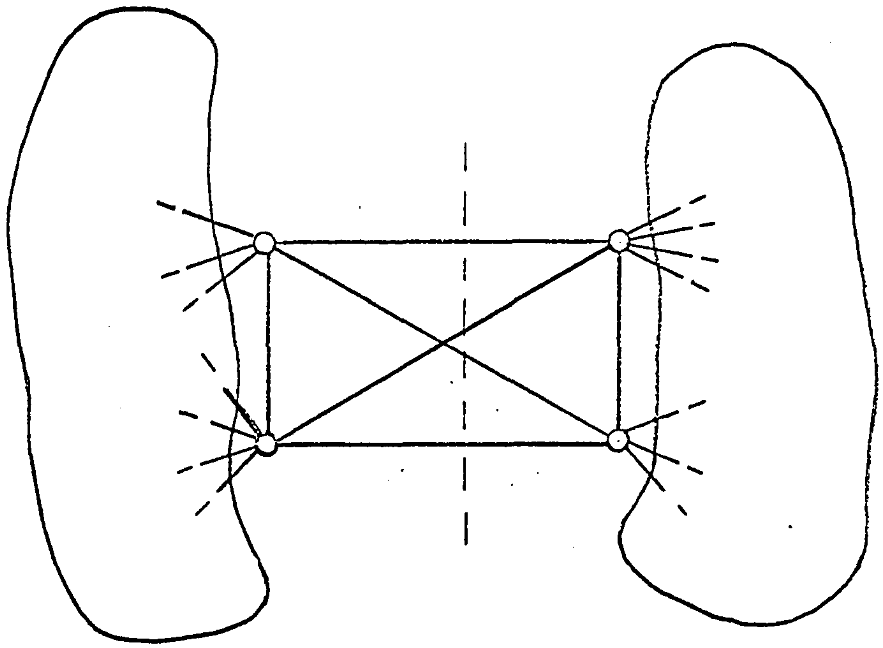
This labelling makes all possible interactions explicit and thus is a considerable aid in finding combinations of operations which interact by having their zero losses in the same matrix addresses. It is seen from the above matrix that adding rows 5 and 6 to row 4 produces one zero and loses two, but both losses are in matrix locations where interactions are possible. Performing the three indicated operations results in the production of three zero pairs and the loss of two, for a net improvement of one zero pair. The corresponding maximally orthogonal fundamental basis set of segs is shown on the graph of Figure 5. Note that the fact that the sets of rows whose members summed to produce zeros were disjoint made this example an exceptionally easy one. Ordinarily considerably more labor would be involved.

The trial and error involved in the steps of procedure I is primarily an effort to locate multiple segs for which $m_i - b_i$ can be less than for a star. Since all possible such segs are relatively few and are readily determined by inspection of a graph, in most cases it is possible to write by inspection the set of segs the procedure will evolve. Similarly, the type of interaction demonstrated in procedure II necessarily has a distinctive geometric appearance, and thus lends itself to being found by inspection. The considerable amount of labor involved in the procedure is then the price of describing a pattern-recognition process algebraically. Moreover, there is a strong correlation between the distinctiveness of the geometric appearance and the amount of improvement in orthogonality that can be realized. Those types of graphs for which an optimum QQ^T matrix has substantially more zeros than does an optimum AA^T matrix tend toward one of the formats shown in Figure 6. The "dumbbell" graph of Figure 6(b) has relatively few multiple segs, but each gives a relatively large improvement over the star it replaces. On the other hand, each multiple seg of the ladder network of Figure 6(a) gains only one zero pair over the star it replaces, but this type of graph has a maximum number of such multiple segs. It is interesting to note that optimum sets of segs for ladder networks derivable from Figure 6(a) by the deletion of any combination of cross-coupling edges will be the same as the set indicated on the Figure except possibly for an inversion of the up-and-down pattern of stars.

If the restriction on interlocking segs is removed, the problem becomes considerably more complex. It appears for several reasons that a



(a)



(b)

Figure 6. Types of graphs for which relatively large improvements in orthogonality can be realized

basis set of segs containing interlocking segs should not have greater orthogonality than some fundamental basis for the same graph, but no general proof of this was found. One mathematically tractable approach is to replace interlocking pairs of segs by non-interlocking segs pairwise. Suppose q_i and q_j are an interlocking pair of segs for a graph. They then partition the vertices of the graph into four sets, shown as sets w_1, w_2, w_3, and w_4 in Figure 7(a). As discussed in the proof of Lemma 19, q_i and q_j may be considered to perform the essential function of partitioning the vertices of the graph so that three vertices, no two of which are in the same set w_k, are partitioned into different sets. The vertex sets assumed to contain one of these vertices are marked with dots in Figure 7. Thus any other pair of segs partitioning the vertices of the graph in a manner that the dot-marked sets w_k are not in the same set can validly replace q_i and q_j in any basis. If the restriction is added that the segs replacing q_i and q_j must be composed of edges contained in the union of q_i and q_j, the resulting replacement of q_i and q_j by a non-interlocking pair of segs will be termed a block decomposition of q_i and q_j. As shown in Figure 7, an interlocking pair of segs has seven block decompositions. The reason for restricting attention to block decompositions is one of mathematical convenience in that the adjacencies of segs formed in this manner may be determined in terms of the adjacencies of the interlocking pair q_i and q_j.

One question which might be logically asked is the following. In view of the relatively large choice of ways to block decompose a pair of interlocking segs, is it possible that one of these decompositions can

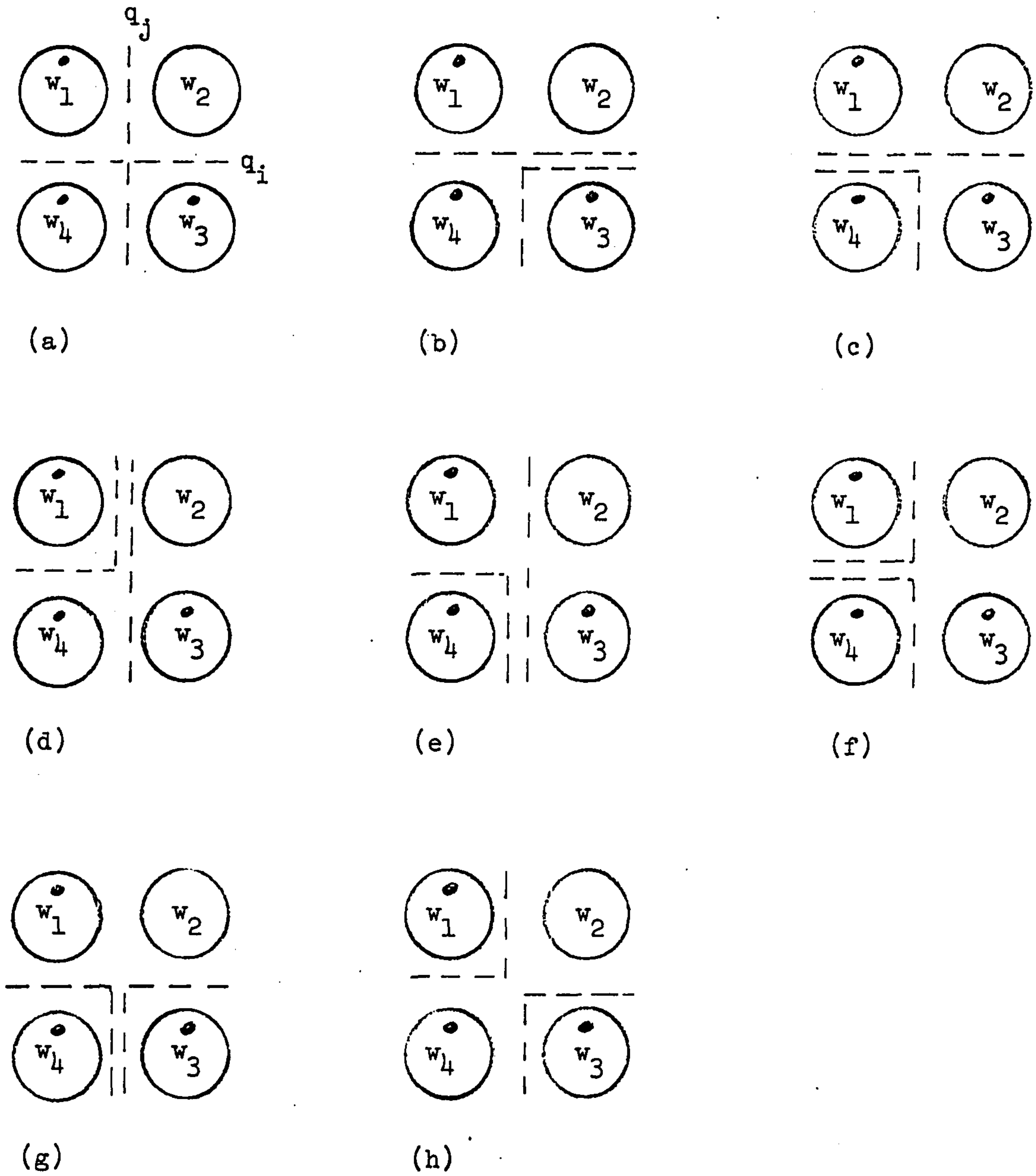


Figure 7. An example of interlocking segs and their seven possible block decompositions

always be found which if performed does not increase the adjacency of a basis? The answer to this question, unfortunately, is no. Conditions for which the decomposition of Figure 7(h) can decrease the orthogonality of a basis are given in the proof of Theorem 19. If conditions under which the remaining six decompositions will increase the adjacency of a basis are computed, it is found that it is possible to conjure up graphs with associated basis sets of segs in which every possible block decomposition of an interlocking pair of segs decreases the orthogonality of the basis. Such a graph and set of segs are shown in Figure 8. This counter-example has the additional property that one of the possible block decompositions, the one shown in Figure 7(h), reduces the set of segs to a fundamental set.

In the special case that every subset of a basis set of segs which is composed entirely of interlocking pairs of segs contains precisely two segs, it is proven in Theorem 19 that a pairwise block decomposition can be performed on such interlocking seg pairs without increasing the adjacency of the basis. Corollary 19 then extends this conclusion to state that in this special case, a maximally orthogonal basis can always be found which is also a fundamental set. This suggests that if a subset composed entirely of interlocking pairs of segs contains m segs, then a block decomposition simultaneously replacing all m segs might always exist which would not increase the adjacency of the basis. If a subset composed entirely of interlocking pairs of segs contains m members, it partitions the vertices of the graph into at least 2m and at most 2^m vertex sets. Suppose the subset partitions the vertices of the graph into not more than $m(m - 1) + 2$ vertex sets in a manner that the relationships

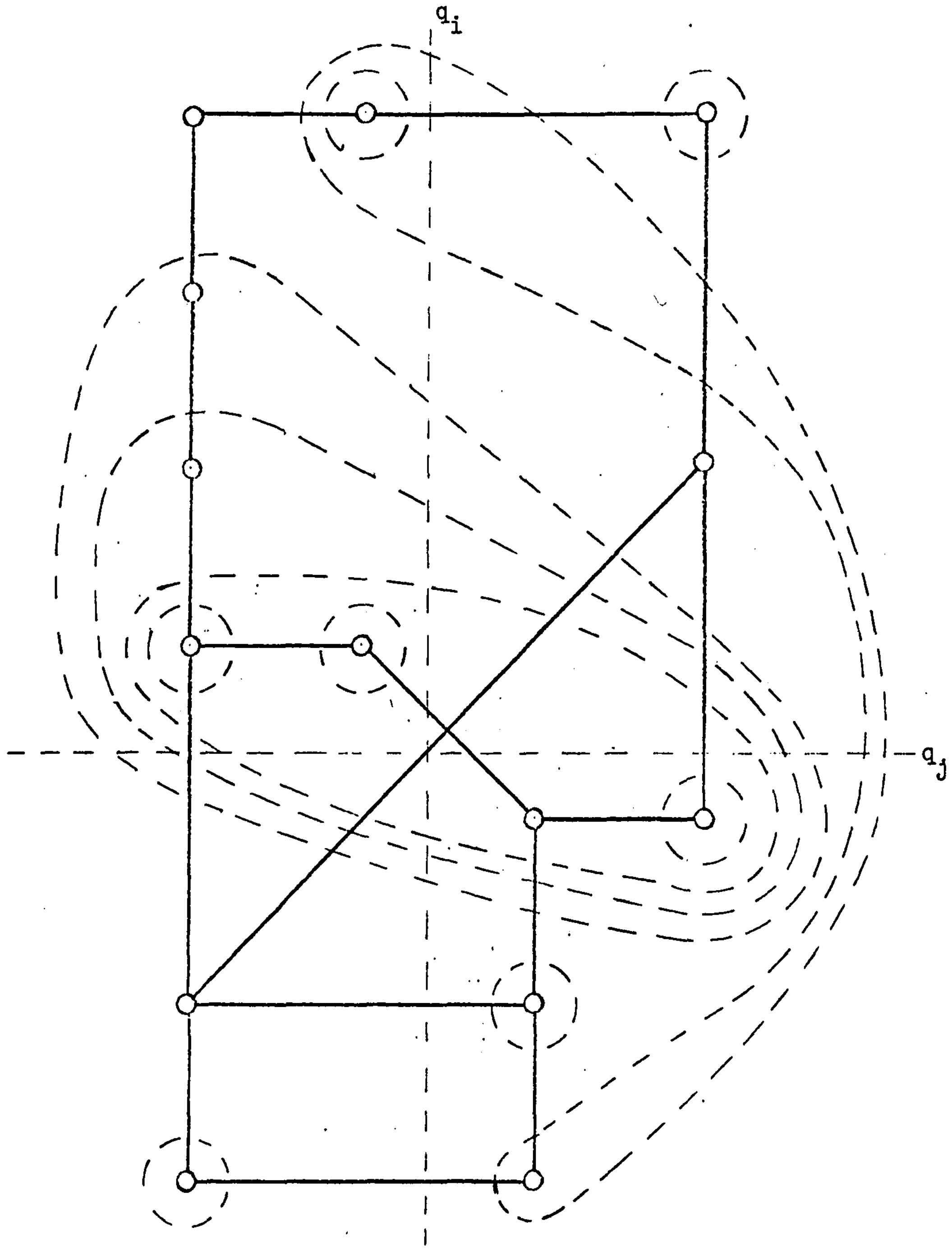


Figure 8. Graph with set of segs in which every possible block decomposition of q_i and q_j decreases the orthogonality

between the segs and the vertex sets can be represented by a two-dimensional Venn diagram. An illustration of such a diagram for the case of five interlocking segs is given in Figure 9. The five circles represent segs and the twenty-two numbered regions represent vertex sets. Since the m interlocking segs must be reducible to stars, there must be m + 1 vertices located in m + 1 different vertex sets in a manner that no two pairs of these vertices are partitioned by only one of the m segs. Any other segs reducible to the same stars can then replace the m interlocking segs. In terms of Figure 9, the circles can always be drawn so that one of the vertices is in region 22, and the remaining five vertices can be distributed in any of the remaining vertex sets subject to the restrictions that no two can be in the same vertex set and each circle (seg) must contain one in its interior. It is convenient to mark each vertex set containing such a vertex with a dot. Since it is assumed that no other segs of the basis to which the m interlocking segs belong is interlocked with any of the m segs, all remaining segs of the basis are such that their edges are all incident upon vertices in a single vertex set. In terms of Figure 9, this means that every other seg can be represented as a closed contour which does not cross over any of the existing lines of the Figure. The process of finding a block decomposition of all m interlocking segs in a manner that the adjacency of the basis is not increased then has a simple interpretation in terms of the Venn diagram. Consider that the line segments between crossovers on the diagram are free to be connected in any fashion at the crossovers. Then if a subset of the line segments can be formed into closed contours that do not intersect

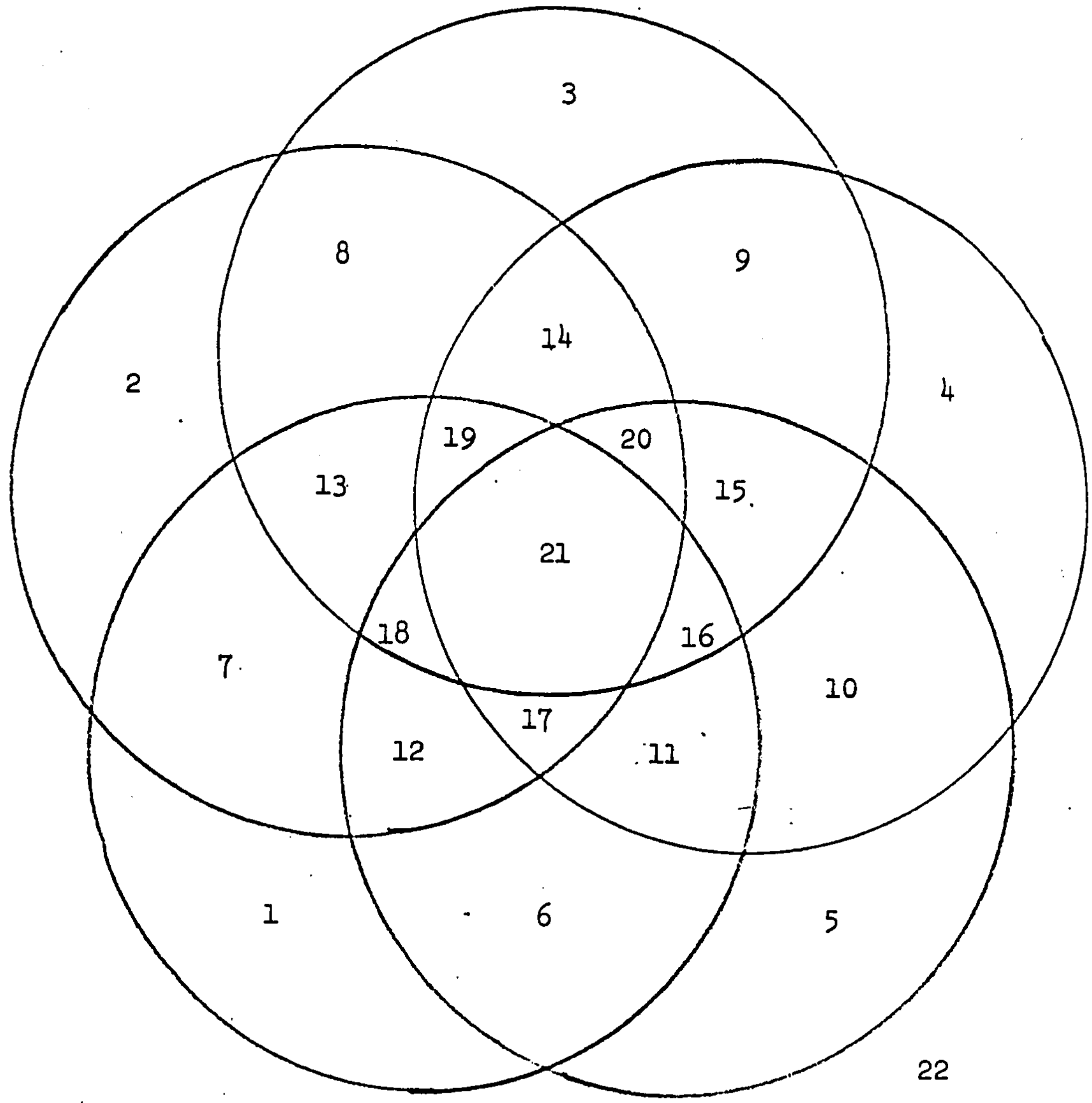
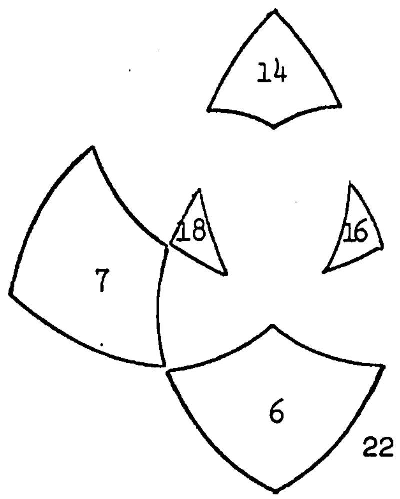


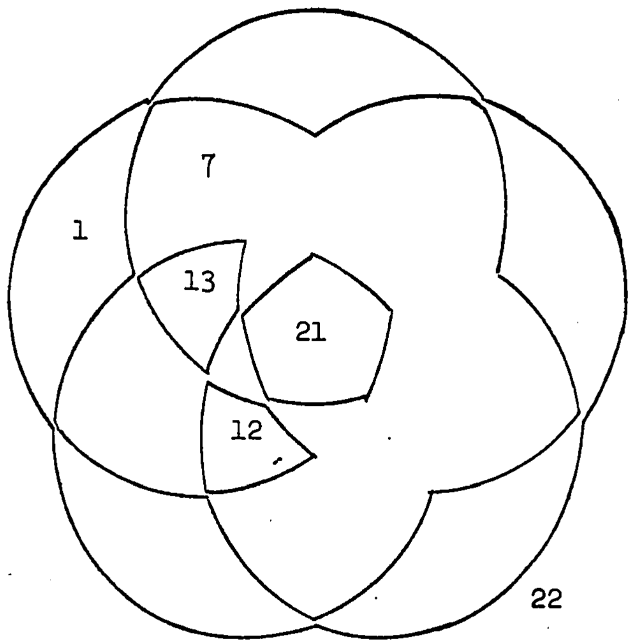
Figure 9. The representation of a set of five interlocking segs as a Venn diagram

themselves and which have no line segments in common in a manner that the dot-marked regions are all separated from one another, the resulting contours will represent the desired decomposition. Two examples of the process are shown in Figure 10. In the first example the dot-marked regions were chosen to be 6, 7, 14, 16, 18, and 22. Since these dot-marked regions are not contiguous, the segs resulting from the decomposition all correspond to single vertex sets (except region 22, which is now combined with all the remaining regions). In the second example the dot-marked regions were chosen to be 1, 7, 12, 13, 21, and 22. Note that in neither example is the desired decomposition unique. The above procedure has been applied to all allowable combinations of dot-marked regions for diagrams representing two, three, four, and five interlocking segs. It was found that with very little practice the desired decomposition could be written by inspection.

In the event the vertex sets formed by the m interlocking segs cannot be represented by a two-dimensional Venn diagram, the theory of a block decomposition is unaffected but the process of actually finding a decomposition is somewhat more difficult. The procedure is as follows. Associate with each seg q_i the corresponding vertex set w_{qi} not containing a designated reference vertex and label each of the 2^m vertex subsets by the indices of the vertex sets containing that subset. Thus vertex set 0 is the set of all vertices not in any w_{qi}, vertex set 1 is in w_{q1} only, vertex set 134 is in w_{q1}, w_{q3}, and w_{q4} only, and so forth. To ensure that a set of m vertex sets represents independent segs it is only necessary to ascertain that all m indices are represented in a manner that no two



(a)



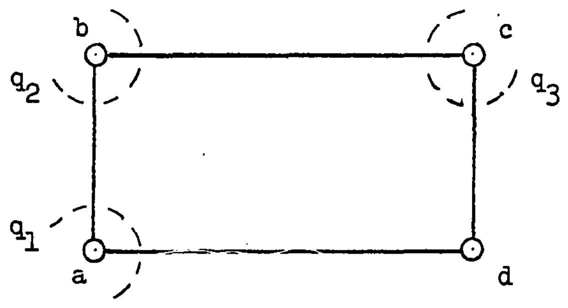
(b)

Figure 10. Two examples of block decomposition of interlocking segs on a Venn diagram

indices are associated with one vertex set only. By analogy with the property of contiguity in a Venn diagram, two vertex sets are said to be contiguous if one can be converted into the other by the addition or deletion of a single index. Thus vertex sets 12 and 13 are not contiguous, but sets 12 and 2 or sets 12 and 125 are contiguous. The requirement that a block decomposition not increase the adjacency of a seg basis can then be stated as follows. If the vertex sets representing any two segs are disjoint, none of the sets representing one seg can be contiguous to any of the sets representing the other, and if the vertex sets representing any two segs have a common set, one group of sets must also contain every set which is contiguous to the common set. For example, if m equals five and the dot-marked vertex sets are chosen to be 1, 2, 13, 14, and 135, a suitable block decomposition consists of sets 2, 14, 135, (13 + 135 + 15 + 35 + 1345 + 1235), and (1 + 3 + 4 + 5 + 13 + 14 + 15 + 34 + 35 + 45 + 123 + 125 + 134 + 135 + 145 + 235 + 345 + 1235 + 1345 + 12345). As in the case of the Venn diagram representation, it has been verified that suitable decompositions exist for all possible combinations of dot-marked vertex sets generated by five or fewer interlocking segs. No way was found to generalize this result to prove that such a decomposition exists for any number of interlocking segs.

In much of the discussion thus far the restriction has been made that the Q and R matrices appearing in the product QR^T are equal. The last topic considered will be concerned with the possible advantages of relaxing this restriction. The first question to be answered involves the number of additional zero entries that can be obtained in an optimum QR^T

matrix over an optimum PP^T for a graph if P, Q, and R are unrestricted seg matrices. Since Q and R both represent bases for the same seg space, one can be converted into the other by means of a nonsingular transformation, so that $QR^T = DRR^T$. Then if the row operations represented by D increase the number of off-diagonal zero entries in RR^T, the corresponding column operations can be performed to obtain additional zero entries. Thus if QR^T is optimum, it can have more zero entries than an optimum PP^T only by virtue of diagonal zero entries. Since the segs of Q and R can always be ordered so that they partition the same n-1 vertex pairs, a diagonal zero entry implies that there must be two equally good non-adjacent choices of segs partitioning the same vertex pair. Thus one single diagonal zero can often be attained. For example, the very simple graph and segs shown in Figure 11 result in QR^T having three zero entries while at most two can be attained if the seg matrices are restricted to be equal. Graphs for which an optimum QR^T contains more than one additional zero than an optimum PP^T matrix are rather rare. Figure 12 is an example in which two additional zeros are gained. It thus appears that the number of zero entries which can be gained by allowing Q and R to be different is sharply limited and probably not worth the loss of symmetry in the coefficient matrix. The real value of allowing Q and R to be different occurs in those situations where one of the matrices is largely or wholly prescribed. Suppose, for example, that R is specified to be a fundamental cut-set matrix based on a particular tree. This type of specification often occurs in situations in which either it is desired to prescribe the independent node variables in terms of which the equations are written or



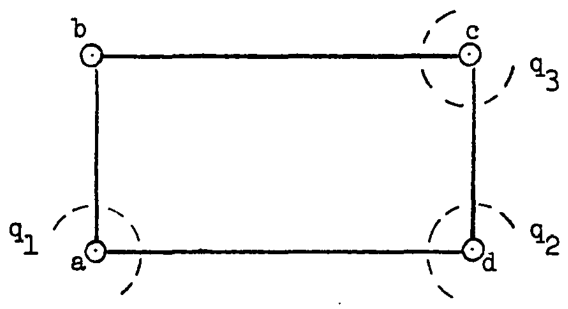
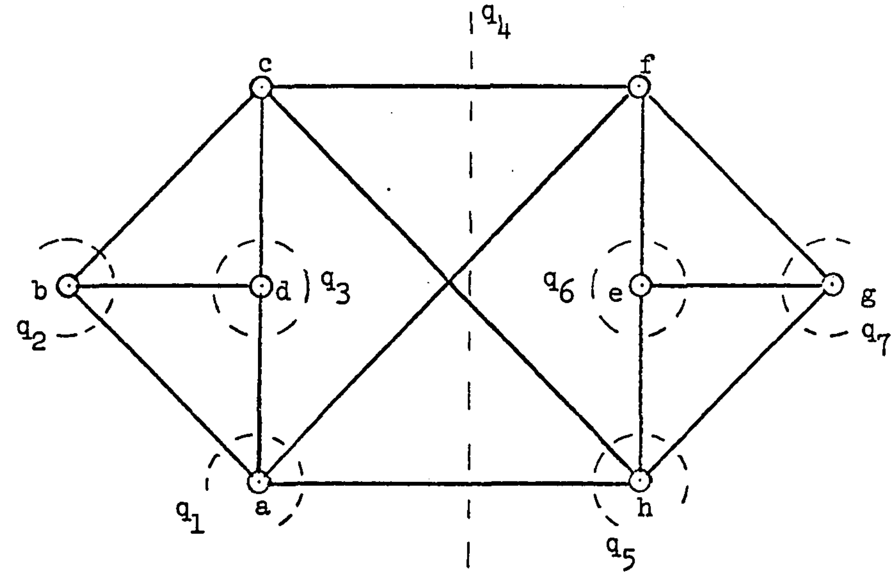


Figure 11. Graph with two sets of segs Q and R for which QR^T gains a diagonal zero entry



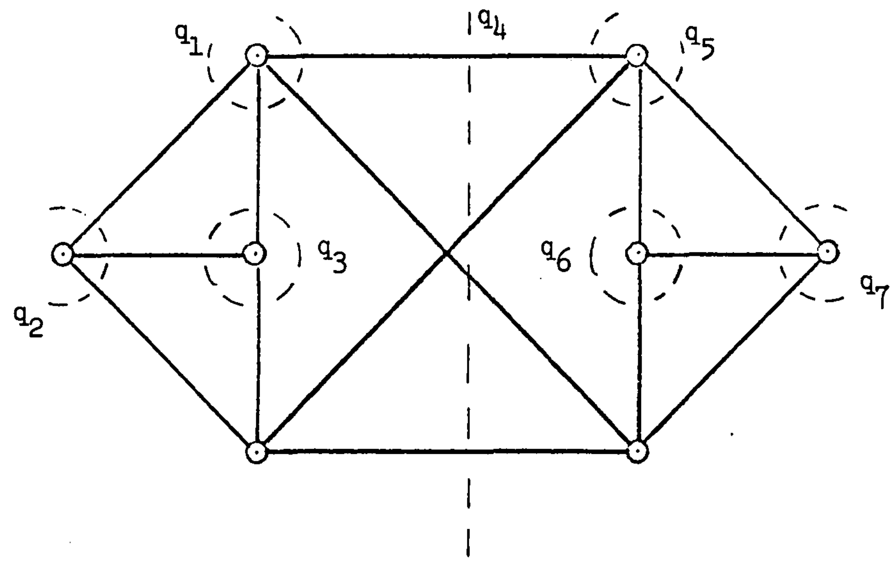


Figure 12. Graph with two sets of segs Q and R for which QR^T gains two diagonal zero entries

it is desired to control the location of symbols representing certain admittances in the coefficient matrix array so as to expedite a subsequent matrix partitioning operation. The specified R may be such that RR^T attains few or none of the possible number of zero entries attainable for the given graph, but another set of segs Q may nearly always be chosen so that QR^T contains most of the attainable zero entries. It is only necessary to base Q on a tree whose branches are, insofar as possible, chords of the previously prescribed tree to ensure that QR^T will be a good if not optimum choice of seg bases.

IV. CONCLUSIONS

The stated purpose of this investigation was to find procedures for formulating nodal equations for an electrical network which would be optimum or nearly so in the sense of being maximally uncoupled. The following conclusions are accordingly noted. An upper bound on the maximum number of zero entries which can be achieved in a coefficient matrix for generalized nodal equations is given by twice the number of non-adjacent vertex pairs in the corresponding graph. This number is equal to the number of zero entries in an indefinite admittance matrix for the network under consideration. In many cases this upper bound cannot be attained. The separable parts of a separable graph represent unrelated problems that are best handled separately, so all conclusions will be assumed to apply to connected nonseparable graphs only. There is very little advantage in choosing Q and R of the coefficient matrix QYR^T to be different seg matrices except in the case that one or the other of them is largely or wholly prescribed. In the latter event the segs in question would ordinarily be a fundamental set based on a tree of the graph (possibly augmented with zero admittance-weight edges). Choosing the other set of segs to be a fundamental set based on another tree whose branches are insofar as possible chords of the first tree ensures that the resulting set of equations will be a good if not optimum choice.

The choice of an incidence set of segs as the set Q will always result in QYQ^T being optimum or nearly so. If Q is restricted to be a fundamental set, algebraic procedures can be followed to find a maximally orthogonal set of segs. These procedures are inefficient in that they require a considerable amount of trial and error, and the improvement in the number of zeros in an optimum QYQ^T coefficient matrix over an optimum AYA^T matrix will seldom if ever be worth the labor involved. A network analyst can easily learn to determine an optimum set of segs simply by inspection of most graphs, however, by learning to recognize distinctive geometric patterns in the graph. Those graphs whose AYA^T matrices admit of substantial improvement tend to approach one of the formats shown in Figure 6.

In the event that Q is not restricted to be a fundamental seg matrix, the problem becomes considerably more complex. It appears virtually certain from several aspects that at least one of the maximally orthogonal basis sets of segs which can be written for any graph will be a fundamental set, but no proof was found for this statement. A purely geometric means was found for reducing interlocking segs to non-interlocking ones without increasing the adjacency of a basis, however it was not verified that this procedure is applicable to all possible cases.

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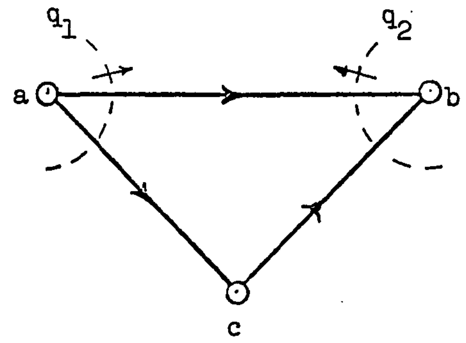
VII. APPENDIX

Linear graph theory not only provides electrical network analysis with firm foundations and versatile manipulative tools, but also allows geometric insight to be applied to algebraic processes. It thus often happens that algebraic processes can be handled by inspection or nearly so in problems studied from a graph-theoretic viewpoint. Such procedures should not be considered mere tricks or short-cuts, but rather effectively making use of one of the major advantages often accompanying the use of this powerful tool. Three procedures of particular interest to the formulation of nodal equations will be given and demonstrated by examples. In all cases, the correspondence between graph and electrical network and the restrictions imposed on networks will be as they have been throughout this investigation. That is, the nodes of the electrical network correspond one-to-one with the vertices of the graph and the graph has an edge adjoining a vertex pair if there is a current path between the corresponding network nodes, and the network is assumed to be reducible to a two-terminal component representation.

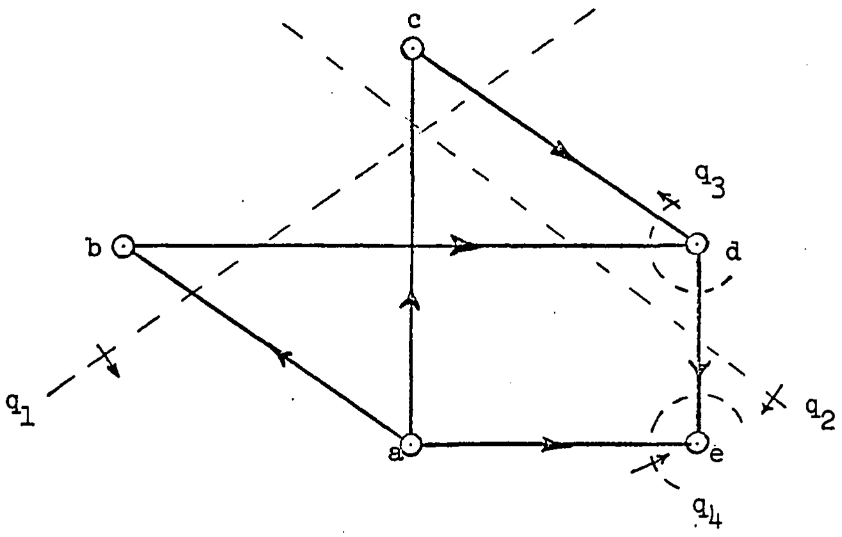
A. Determining the Node Variables Associated With a Given Set of Segs

A set of generalized nodal equations may be written in matrix form as $PYQ^TV_n = PI$ where P and Q are seg matrices, Y is an element admittance matrix, I is a source matrix, and V_n is the node variable matrix. The elements of V_n are node-pair voltages or sums of node-pair voltages and are determined by seg matrix Q only. Q and V_n enter into a set of nodal equations via what is known as the node transformation, $Q^TV_n = V_b$, where V_b is the column matrix whose elements are the node-pair voltages associated with each branch of the electrical network. In order that each possible node-pair voltage of a network correspond to an edge of the graph of the network, it is convenient to assume that a given graph is fully connected, that is, that every vertex pair of the graph is connected by an edge. This is no restriction, however, because any graph can be made fully connected by augmenting it with zero admittance-weight edges without affecting any sets of nodal equations based on that graph. For a graph with n vertices the matrix V_b is then a column matrix whose $\frac{n(n-1)}{2}$ rows represent every possible node-pair voltage of the network corresponding to the graph. But only n - 1 of these voltages may be independently specified, and all remaining voltages must then be expressible as linear combinations of the specified voltages. The n - 1 rows of column matrix V_n are such an independent set, and the $\frac{n(n-1)}{2}$ rows of Q^T each define the particular linear combination of the elements of V_n which is equal to the node-pair voltage in the corresponding row of V_b. In a pictorial representation of segs on a geometric graph, each nonzero entry in Q^T corresponds to a seg unavoidably crossing over an edge. The particular linear combination of node variables forming any particular node-pair voltage is thus explicitly presented schematically. It is this fact that allows the rapid determination of node variables from a sketch of segs on a graph.

As examples of the technique, sets of node variables will be found for the graphs of Figure 13. The symbol v_i will be used both to denote the i^{th} vertex and to represent the potential of the corresponding network node with respect to any arbitrary reference. The node variable associated



(a)



(b)

Figure 13. Examples used to demonstrate the determination of node variables

with a seg q_i will be denoted as v_{qi}. By convention, the edge orientation is that of the assumed direction of conventional current flow in the corresponding network, and so is useful for describing a network branch voltage in terms of a node-pair voltage. The edge orientations do not enter into the determination of node variables, however. The orientations associated with segs are arbitrary and serve to determine the signs of entries in the seg matrix. In Figure 13(a) the voltage across branch ac of the circuit will be $v_a - v_c$, and since this node-pair voltage appears in only one seg, this must be the node variable associated with that seg. The orientation assigned the seg is in the direction from a to c, so $v_{q1} = v_a - v_c$. Similarly, edge cb is cut by seg q_2 only, so $v_{q2} = v_b - v_c$. In the event it had not been noticed that q_2 was the only seg separating vertex pair cb, one could also utilize vertex pair ab to write $v_a - v_b = v_{q1} - v_{q2}$. Substituting the known value of v_{q1} and rearranging results in $v_{q2} = (v_a - v_c) - (v_a - v_b) = v_b - v_c$ as before. The entire set of node variables will be node-pair voltages if and only if every seg is the only seg separating some vertex pair, that is, if the set of segs is a fundamental set.

An example of a non-fundamental set of segs is given in Figure 13(b). Seg q_1 is found to be the only seg separating vertices b and a and is oriented from b to a, so $v_{q1} = v_b - v_a$. Seg q_2 is the only seg separating vertices c and b and is oriented from c to b, so $v_{q2} = v_c - v_b$. Similarly, $v_{q4} = v_a - v_e$, but q_3 is not the only seg separating any vertex pair. We choose any vertex pair separated by q_3, say vertices d and a, and write $v_d - v_a = v_{q3} + v_{q2}$. Substituting the known value for v_{q2} and rearranging results in $v_{q3} = (v_d - v_a) + (v_b - v_c)$. If vertices c and d had been chosen, the equation to be solved would have been $v_d - v_c = v_{q3} - v_{q1}$, which would lead to the same result as before for v_{q3}. If vertices d and e had been chosen the equation to be solved would have involved three node variables but again would lead to the same result.

B. Determining Whether Or Not a Given Set of Segs Is Independent

The determination of whether or not a given set of segs is independent may be rapidly accomplished by reducing all the segs to stars, and because the independence of a set of segs does not depend on the edge structure of a graph, it is convenient to work entirely with vertex sets. It is entirely possible that an independent set of segs for a directed graph will cease to be independent if the edge orientations are removed, but this circumstance is sufficiently rare that the computational advantages of modulo 2 algebra justify attempting this approach first. If the modulo 2 reduction process results in an independent set of stars (one for each vertex except for the reference), the original set of segs was independent. If in the course of the reduction two segs are made equal then either the set of segs is not independent or the set of segs was one which would cease to be independent if edge orientations were removed. In the latter case one can either use ordinary algebra to effect the reduction process or simply inspect the geometric pattern formed by the segs on the graph. If a reference vertex is selected and all segs are sketched on the geometric graph as closed contours not containing the reference vertex, then all sets of segs which cease to be independent when edge orientations are removed will have a subset giving the appearance of a closed chain.

As an example of the process of determining whether a given set of segs is independent, consider the graph and set of segs given in Figure 14(a). If vertex c is arbitrarily chosen to be reference and all segs represented by the vertex set they generate which does not contain c, one possible modulo 2 reduction process proceeds as given below.

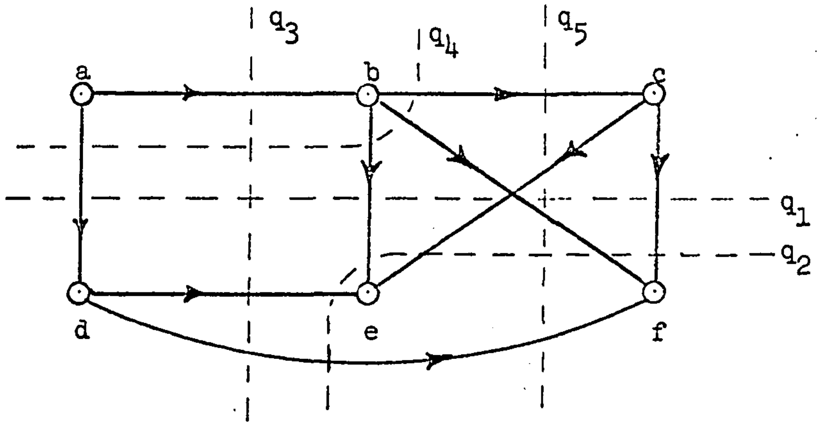
q_1 = def	d	d	d	d	d
q_2 = ef	ef	ef	ef	ef	f
q_3 = ad	ad	a	a	a	a
q_4 = ab	ab	ab	b	b	b
q_5 = abde	abde	abe	abe	e	e

The first column is the initial set of segs. The second column is the result of adding (modulo 2) row 2 to row 1 in the first column. Then adding row 1 of the second column to rows 3 and 5 of that column results in the vertex sets shown in the third column. The procedure is continued until an independent set of stars is achieved, thus the initial set of segs is independent.

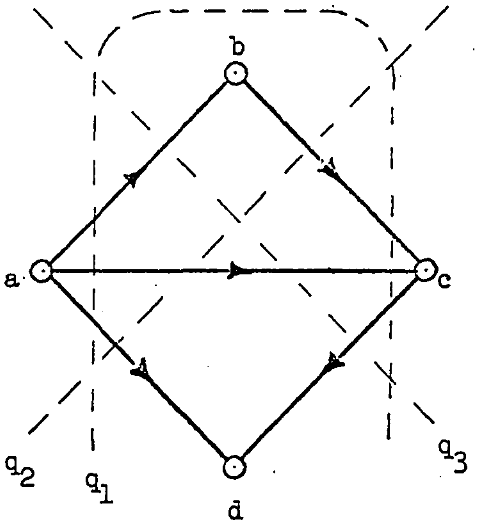
In the example of Figure 14(b), one modulo 2 reduction process is as follows. Vertex d has arbitrarily been chosen to be reference.

q_1 = ac	ac
q_2 = ab	bc
q_3 = bc	bc

Here the modulo 2 reduction process has failed. If the segs are sketched as closed contours not containing the reference vertex, it is seen that they do indeed form a closed chain. It is thus still possible that this set of segs is independent. To verify that it is, an ordinary algebra



(a)



(b)

Figure 14. Examples used to demonstrate the determination of whether or not a given set of segs is independent

reduction process is performed as given below.

$q_1 = a + c$	$a + c$	$a + c$	$a + c$	a
$q_2 = a + b$	$c - b$	$c - b$	$c - b$	b
$q_3 = b + c$	$b + c$	$2c$	c	c

The reduction process successfully leads to an independent set of stars, hence the original set of segs is an independent set.

C. Writing the QYR^T Coefficient Matrix By Inspection

If the segs of seg sets Q and R are sketched on a graph, the elements of the matrix product QYR^T can be determined directly by inspection if the electrical network and graph are as discussed in the beginning of this Appendix. The procedure simply amounts to performing the indicated matrix operations of QYR^T visually, and is as follows. Let Y_{ij} be an element of QYR^T and let q_i and r_j be the segs represented by the i^{th} and j^{th} rows of Q and R respectively. Then Y_{ij} equals the algebraic sum of admittance weights of edges common to q_i and r_j, where each such edge admittance weight is used with its given sign if q_i and r_j are similarly oriented with respect to that edge and with its sign reversed if q_i and r_j are oppositely oriented with respect to that edge.

The graph and sets of segs given in Figure 15 will be used to illustrate the procedure. Segs q_1 and r_1 have one edge, bc, in common and are similarly oriented with respect to bc, so $Y_{11} = y_{bc}$ where y_{bc} is the admittance weight of edge bc. Segs q_1 and r_2 have edges bc and ac in common and are oppositely oriented with respect to both, so

$$Y_{12} = - y_{bc} - y_{ac}.$$

The continuation of this process results in the matrix below.

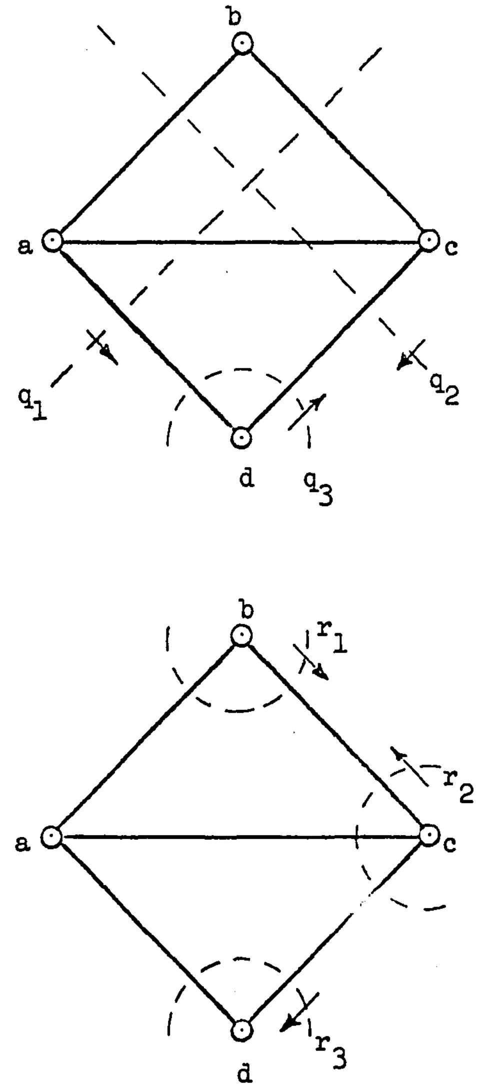


Figure 15. Example used to demonstrate the determination of QYR^T

$$QYR^T = \begin{bmatrix} (y_{bc}) & (-y_{bc} - y_{ac}) & (y_{ad}) \\ (y_{ab}) & (y_{ac} - y_{cd}) & (y_{cd}) \\ 0 & (-y_{cd}) & (-y_{ac} - y_{cd}) \end{bmatrix}$$

In the event that Q and R are chosen to be equal, the i^{th} diagonal element of QYQ^T will consist of simply the sum of the weight factors of the edges of q_i.